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Modeling, Simulation, and Applications of Fractional Partial Differential Equations

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by

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Abstract

The Black-Scholes model is commonly used to track the price of European options with respect to maturity in many financial markets. This model degenerates into a partial differential equation that relates the European-style option price to the underlying price and time of expiry. Black-Scholes assumes that underlying prices satisfy a geometric Brownian motion.

After the U.S. stock market crash of 1987, this assumption becomes inaccurate as it fails to represent the behavior of S&P 500 European vanilla option prices. Specifically, under the measure of moneyness, the volatility smirk does not flatten out and the resulting conditional return distribution does not converge to normality. Recent academic literature have proposed readjusted financial models to account for the shortcomings of Black-Scholes, none which successfully have combined infinite return moments and finite price moments.

To reduce the effects of these consequences and to incorporate the additional moment conditions, we assume that the underlying satisfy a Levy $\alpha-$stable motion. Under this assumption, we will derive the Finite Moment Log Stable (FMLS) model and its respective fractional partial differential equation counterpart. Then, we will solve the Black-Scholes equation under FMLS by using the standard finite difference method and a finite volume scheme that significantly reduces the computational and storage cost in comparison. Lastly, we will perform a numerical simulation of our methods by using recent financial data in the S&P 500 market acquired within a one-year time frame to compare the performance of these methods.

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Introduction

Constructed by Fischer Black and Myron Scholes, the Black-Scholes model is a financial instrument designed to analyze the behavior of European option prices under changes of maturity dates and the stochastic behavior of asset prices. Contrary to most financial markets, this model assumes that asset prices are distributed normally and thus, satisfy a geometric Brownian motion. Realistically, the volatility and interest rate are varying and randomly behaved. To simplify the construction of the Black-Scholes model, we assume that these parameters are constant and known. Our choice of these parameters will be highly dependent on recent trends of the current market. [2] Assuming the conditions previously mentioned, there exists a partial differential equation that captures the behavior of European option prices satisfying the Black-Scholes model. In most financial and mathematics literature, this equation is commonly known as the Black-Scholes equation. The derivation of Black-Scholes can be reproduced with stochastic analysis using applications of finance. One can also derive the closed form solution of the Black-Scholes model by solving the traditional diffusion model under specific changes of variables and Fourier transforms.

However, the Black-Scholes model fails to capture the behavior of option prices under rare and extreme circumstances as these assumptions are not applicable realistically. This is shown immediately after the U.S. stock market crash of 1987, where numerous academic resources have identified a consistent pattern found in the S&P 500 options market. At a given maturity date, the Black-Scholes model implies that, with respect to the strike price, the volatilities for out-of-the-money puts are much higher than out-of-the-money calls. In finance, this phenomenon is known as volatility smirk. Under the measure of moneyness, the implied volatility smirk does not flatten out as maturity increases. Namely, one can observe that the downward slope of the volatility smirk corresponds to the asymmetry of the return distribution and the positive curvature of the smirk correlates with the existence of leptokurtosis. [3] Therefore, the conditional return distribution does not converge to normality, the central limit theorem does not hold, and the assumption that asset prices satisfy a geometric Brownian motion cannot be made in these rare situations. This calls for a different model that accurately tracks this recent behavior of the S&P 500 options market.

In doing so, we require the construction of a variant of the Black-Scholes model to generalize the behavior of option pricing in various financial markets. We need to model returns by considering an α-stable motion with maximum negative skewness where α is the tail index satisfying $0 < \alpha \leq 2$. The result of this model will combine infinite return moments and finite price moments. These adjustments are necessary to accurately reflect the observed behavior of S&P 500 option prices documented after the market crash. When $\alpha = 2$, the α-stable motion becomes the standard Brownian motion, and the resulting model constructed from the α-stable motion degenerates into the standard Black-Scholes model. Setting $\alpha < 2$ yields many interesting properties that lie in the foundation of the study of fractional calculus. Similarly to the Black-Scholes equation, there exists a fractional counterpart that reflects the generic behavior of European vanilla option pricing. We begin Chapter 1 by deriving the classical Black-Scholes equation using methods in stochastic calculus and its applications in finance. Then, we will gather an intuitive understanding of fractional derivatives to study the traditional diffusion model and its fractional counterpart. We will construct a generalization of the Black-Scholes model that requires the theory of stable random variables to accurately track the behavior of European option pricing. Under this derived model, we will finally derive and analyze the Black-Scholes equation under this model.

After this derivation, observing the behavior of the price of a European vanilla option under this model and determining the respective analytic solution becomes a natural question. In doing this, we study the closed form of the Levy density function, which can be written in terms of the Fox H-function. When $\alpha = 2$, the Levy density function becomes the probability density function of the standard Gaussian distribution. This function plays a crucial role in determining the analytic solution of the price of European vanilla options. The properties that arise from the analytic form of the Fractional Black-Scholes equation and its consistencies with the Black-Scholes formula become other questions of interest. In Chapter 2, we derive this analytic form using the Fox H-function and propose associated theorems that emphasize the properties of the closed form corresponding to the Black-Scholes formula. We prove these theorems to provide mathematical justification for the observations documented in many options markets. We will also analyze the asymptotic behavior of the closed form solution with respect to the log price and show that this solution satisfies the put-call parity.

Many numerical methods have been proposed to solve partial differential equations with fractional order. Among many, the finite difference method is known to be an effective method that uses numerical values evaluated at specific nodes to approximate intermediate values. This is achieved by partitioning bounded intervals into finite increments and using the finite difference approximation of the derivatives to construct a recursion that defines the behavior of the function. To use the finite difference method, we are required to truncate the unbounded interval to the finite interval $[-\varphi, \varphi]$ for some constant φ. The constant φ depends on the maximum price of the asset and can be readjusted with respect to the necessary conditions of the market. In Chapter 3, we recall the definition of a fractional derivative in terms of Grunwald weights from Chapter 1. We will show that a shift on the weights is required to yield a consistent and unconditionally stable solution of the finite difference method. Under the assumption that the log price is contained in some bounded interval, this will result in a recursion that relates numerical values of consecutive nodes. This recursion can be reverted into a matrix equation which can be solved using standard Gaussian elimination.

Determining more efficient methods has been an area of interest in the study of numerical analysis. We propose a few approaches in using efficient finite difference methods which reduces the computational cost from $O(m^3)$ to $O(m \log^2 m)$ and the storage cost from $O(m^2)$ to $O(m)$ where m represents the number of unknowns. [13] Contrary to the standard method of Gaussian elimination, this reduction of cost can be achieved by proposing banded coefficient matrices that approximate the default coefficient matrix with reduced storage cost. In the first approach, we require the boundary conditions to take the value of zero at the endpoints. We circumvent this issue by using the Taylor series expansion of the closed form to determine two points which will take the same values. In the second method, we require the partial differential equation to be rewritten conservatively. Both methods will generate matrix equations that can be solved using banded coefficient matrices. In Chapter 4, we will first manipulate the current problem so that the necessary conditions for both methods hold. After doing this, we will apply these methods for the fractional counterpart of the Black-Scholes equation to determine the behavior of European vanilla option pricing under the derived Finite Moment Log Stable process defined in Chapter 2. Lastly, we will implement a Fast Conjugate Gradient Squared Method to accelerate the performance of the finite volume scheme by reducing the storage cost.

In the last chapter, we will perform a numerical simulation of the finite difference methods discussed throughout this thesis by using data observed in the most recent one year time period of the S&P 500 options market. We will simulate the prices in a graph to observe the effects of European put options as α varies. We will also discuss the accuracy of the finite volume scheme and emphasize the low CPU time required to implement the method under various approaches. We will finally make observations on the data acquired in our simulation to support the theory developed in the previous chapters.

Chapter 1

Model Problem

In this chapter, we will derive the Black-Scholes equation using financial applications and stochastic calculus. This will require many assumptions to simplify the derivation of the model. Under these assumptions, we will discuss why the Black-Scholes model fails to capture the recent behavior of the S&P options market. We will proceed to acquire an understanding of fractional derivatives by analyzing the fractional counterparts of the traditional diffusion model and the central limit theorem. We turn to a generalized class of stochastic processes that mimics many similar characteristics of the conditional return distribution in the S&P options market. Specifically, we will define and study properties of α-stable motion and derive a readjusted model highlighting the inconsistencies of the traditional Black-Scholes model. Then we will conclude this chapter by deriving the Black-Scholes equation under the FMLS model using our understanding of fractional calculus and α-stable motion.

1.1 Preliminaries

In 1826-1827, Robert Brown conducted a study on the movement of particles from pollen grains suspended in water. He noted that these particles move in an irregular manner and that the motions of two distinct particles appear to be independent. These observations are similarly observed in fluctuations of stock prices by French mathematician Louis Bachelier in the early 1900s.

In 1905, Albert Einstein provided a mathematical explanation of the phenomena. Consider a tube of clear water and inject a unit amount of ink at time $t = 0$ and at location $x = 0$. Define $u(x,t)$ to be the density of ink particles at location x and time t. Let $f(y,\tau)$ be the probability density function of an ink particle moving from distance x to $x+y$ in relatively small time τ. Then, by Taylor Series expansion of $u(x-y,t)$, we have:

$$\begin{aligned}
u(x,t+\tau) &= \int_{-\infty}^{\infty} u(x-y,t)f(y,\tau)dy \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n u(x,t)}{\partial x^n}(x+y)^n \right) f(y,\tau)dy \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n u(x,t)}{\partial x^n} y^n \right) f(y,\tau)dy + O(\tau x) \\
&= \int_{-\infty}^{\infty} \left(u(x,t) - \frac{\partial u(x,t)}{\partial x} y + \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} y^2 \right) f(y,\tau)dy + O(\tau) \\
&= u(x,t) \int_{-\infty}^{\infty} f(y,\tau)dy - \frac{\partial u(x,t)}{\partial x} \int_{-\infty}^{\infty} y f(y,\tau)dy \\
&\quad + \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} \int_{-\infty}^{\infty} y^2 f(y,\tau)dy + O(\tau)
\end{aligned}$$

Since f is a probability density function, for any small τ, we have $\int_{-\infty}^{\infty} f(y,\tau)dy = 1$. Given $f(-y,\tau) = f(y,\tau)$, it follows that $\int_{-\infty}^{\infty} y f(y,\tau)dy = 0$. We assume that the variance of f is linear in τ; that is, for some constant $\sigma^2 > 0$, we have,

$$\int_{-\infty}^{\infty} y^2 f(y,\tau)dy = \sigma^2 \tau.$$

Continuing from our calculations, we have,

$$\begin{aligned}
u(x,t+\tau) &= u(x,t) \int_{-\infty}^{\infty} f(y,\tau)dy - \frac{\partial u(x,t)}{\partial x} \int_{-\infty}^{\infty} y f(y,\tau)dy \\
&\quad + \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} \int_{-\infty}^{\infty} y^2 f(y,\tau)dy + O(\tau) \\
&= u(x,t) + \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} \sigma^2 \tau + O(\tau)
\end{aligned}$$

Equivalently,

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} \sigma^2 + O(\tau).$$

Letting $\tau \to 0$ yields,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}.$$

The above partial differential equation is commonly known as the **traditional diffusion model**. This model has the following closed-form solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{x^2}{2\sigma^2 t}\right).$$

This diffusion model and its solution can be mathematically justified using random walks and the Laplace-DeMoivre Theorem. A rigorous proof of the derivation and the solution of this model can be found in Appendix A.

Later in this thesis, we will provide a second proof of the diffusion model using Fourier transforms. We can extend this alternate proof to create a diffusion equation involving fractional derivatives to gather intuition of super-diffusion. Specifically, we will generalize our understanding of diffusion to develop an understanding of fractional derivatives and their properties. This understanding is crucial to the derivation of the Black-Scholes equation under the FMLS model.

1.2 Derivation and Consequences of Black-Scholes Model

In this section, we will derive and discuss the consequences of the Black-Scholes model. More information can be referred to Roberts. [11] Let $C(S,t)$ be the price of a European call option for asset price S and time t. We assume that $C(S,t)$ varies smoothly with respect to t and S so that its partial derivatives $\frac{\partial C}{\partial t}$, $\frac{\partial C}{\partial S}$, and $\frac{\partial^2 C}{\partial S^2}$ are well defined and smoothly varying.

We wish to construct a risk-free portfolio of one call option and φ units of assets. Let Π be the value of this portfolio. Since $\Pi = -C(S,t) + \varphi S$ is a function of stochastic asset value S, Π is an Ito process. To observe the behavior of Π over a small incremental change of time dt, we must consider the stochastic differential of Π, namely $d\Pi = -dC + \varphi dS$.

The key assumption of the Black-Scholes model is that asset price satisfies a geometric Brownian motion. Therefore, we have

$$\frac{dS}{S} = \alpha dt + \beta dW_t$$

where W_t represents Brownian motion with respect to time t, and α and β represents the stock drift and stock volatility respectively. Equivalently,

$$dS = \alpha S dt + \beta S dW_t.$$

We will use Ito's Chain Rule (Theorem B.1) to derive the Black-Scholes equation. The proof of this result can be found in Appendix B. Applying Ito's Chain Rule with $Y = -C$, $x = S$, $\mu = \alpha S$, and $\sigma = \beta S$ yields the following:

$$\begin{aligned} d\Pi &= \left(\left(-\frac{\partial C}{\partial t} - \alpha S \frac{\partial C}{\partial S} - \frac{1}{2}\beta^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt - \beta S \frac{\partial C}{\partial S} dW_t\right) + \varphi\left(\alpha S dt + \beta S dW_t\right) \\ &= \left(-\frac{\partial C}{\partial t} - \alpha S \frac{\partial C}{\partial S} - \frac{1}{2}\beta^2 S^2 \frac{\partial^2 C}{\partial S^2} + \alpha S \varphi\right) dt + \beta S \left(-\frac{\partial C}{\partial S} + \varphi\right) dW_t \end{aligned}$$

Under the assumption that the portfolio is risk-free, we require the volatility of $d\Pi$ to be zero. This implies that to guarantee this assumption, we need $\varphi = \frac{\partial C}{\partial S}$. In doing so, we obtain the following:

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt.$$

Furthermore, the risk-free assumption implies that the returns of the portfolio equal to the returns of the investments made in bonds. With the value of the portfolio being $-C + \frac{\partial C}{\partial S} S$ and interest rate r, we have

$$d\Pi = r\left(-C + \frac{\partial C}{\partial S} S\right) dt.$$

Equating the coefficients and rearranging terms yields the renowned **Black-Scholes equation** for the value $C(S, t)$ of the call option,

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\beta^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

With the boundary conditions that for $0 \le t \le T$ and $0 < S < \infty$, $C(S,t) \sim S$ as $S \to \infty$, and $C(S,T) = \max\{S - K, 0\}$ for strike price K, the solution of the Black-Scholes equation, known as the **Black-Scholes Formula**, is provided below:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where,

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}s^2\right) ds$$

And,

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}}$$
$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}} = d_1 - \beta\sqrt{T-t}$$

Remark: Note that:

$$\frac{d_1^2 - d_2^2}{2} = \frac{1}{2}(d_1 + d_2)(d_1 - d_2) = \frac{1}{2}\beta\sqrt{T-t}\left(2d_1 - \beta\sqrt{T-t}\right) = \log(S/K) + r(T-t).$$

Therefore,

$$\frac{S}{K}e^{r(T-t)} = \frac{N'(d_2)}{N'(d_1)} \iff SN'(d_1) = Ke^{r(T-t)}N'(d_2).$$

The derivation of the Black-Scholes formula requires the correct substitutions for many variables to transform the Black-Scholes model into the traditional diffusion model. More details on the derivation of the Black-Scholes formula can be found in Appendix C.

The Black-Scholes equation emphasizes the relationship between many observable parameters and the stock volatility β; specifically by using the Black-Scholes Model, β can be determined by the observable call option value $C(S,t)$, the asset price S, the time to maturity t, and the risk-free interest r.

Note that by fixing these observable parameters in the Black-Scholes formula, we have:

$$\begin{aligned}\frac{\partial C}{\partial \beta} &= SN'(d_1)\frac{\partial d_1}{\partial \beta} - Ke^{r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \beta} \\ &= \sqrt{T-t}\left(Ke^{r(T-t)}N'(d_2)\right) + \frac{\partial d_1}{\partial \beta}\left(SN'(d_1) - Ke^{r(T-t)}N'(d_2)\right) \\ &= S\sqrt{T-t}N'(d_1) \\ &= S\sqrt{\frac{T-t}{2\pi}}\exp\left(-\frac{d_1^2}{2}\right) > 0.\end{aligned}$$

By the above calculations, we can deduce that the value of the call option is a monotonic increasing function of its implied volatility. This enables us to compute the volatility for various call options with respect to different asset prices and maturities. Due to monotonicity, the implied volatility can be interpreted as a re-scaling of option prices necessary to analyze its behavior with respect to the observable parameters. It is known that in most financial markets, the implied volatility flattens out as maturity increases, and thus, the conditional return distribution should converge to normality. In finance, this behavior is renowned as the **volatility smirk**.

However, trends following the 1987 U.S. stock market crash suggest otherwise. Contrary to the implications of the Black-Scholes model, it is well documented that according to the S&P 500 index options market, the implied volatility does not flatten out as maturity increases. This is made apparent when implied volatility is observed with respect to maturity and financial measure moneyness $d := \frac{\log(S/K)}{C\sqrt{T-t}}$, for some constant C. Moneyness approximately determines the number of standard deviations that the strike is apart from the forward price. It can be observed that the downward slope of the observed smirk corresponds to asymmetry and the positive curvature of this smirk corresponds to the fat tails in the conditional return distribution. Due to these observations, the Black-Scholes model fails to capture the true behavior of the recent trends of option pricing.

This motivates us to study the effects of the return distribution when one changes the stochastic behavior of asset prices. Recall that for the Black-Scholes model, we assumed the following:

- The value of the call option varies smoothly.
- The asset price satisfies a geometric Brownian motion and returns are lognormally distributed.
- The portfolio is risk-free and the model assumes constant volatility.

These assumptions are consequences of the validity of the central limit theorem. We wish to readjust our model to compensate for the shortcomings of Black-Scholes. In most financial literature, it has been proposed to assume that asset prices satisfy a more generic class of stochastic processes called Levy $\alpha-$stable motion. More discussion on this class of stochastic processes will be presented later in this chapter. One can construct a variant of the Black-Scholes equation that accurately reflects the behavior of implied volatility over maturity in many financial markets.

In the next section, we will reprove the diffusion model using Fourier transforms and highlight the correlation of this model to the central limit theorem. We will discuss the proof conceptually by using the theory of random walks and extend this proof to derive the fractional diffusion model. Then, we will briefly discuss the correlation of the fractional diffusion model to the conceptual ideas of super-diffusion.

1.3 Traditional and Fractional Diffusion Models

Recall that the traditional diffusion model is governed by the partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}$$

and has the closed-form solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right).$$

We will reconstruct a proof of the diffusion model using Fourier transforms. This will require a definition and an example that will be useful later in this thesis.

Definition 1.1. *Let Y be a random variable with probability density function $f(x)$, differentiable and bounded on the interval $[a,b]$ with $f^{(n)}(a) = f^{(n)}(b)$ for all n. We define the* ***Fourier transform*** *of $f(x)$ to be the following:*

$$\hat{f}(k) = \mathcal{F}\{f(x)\} := \mathbb{E}[e^{-ikY}] = \int_a^b e^{-ikx} f(x)dx.$$

Example 1.1. We will attempt to compute the Fourier transform of $f^{(n)}(x)$. We claim that $\hat{f}^{(n)}(x) = (ik)^n \hat{f}(x)$ and proceed to prove the claim with induction. By definition and applying integration by parts on $f'(x)$, we have,

$$\begin{aligned}
\hat{f}'(k) &= \int_a^b e^{-ikx} f'(x)dx \\
&= [e^{-ikx} f(x)]_a^b + ik \int_a^b e^{-ikx} f(x)dx \\
&= ik\hat{f}(x).
\end{aligned}$$

Now assume $\hat{f}^{(m)}(x) = (ik)^m \hat{f}(x)$. Then, similarly,

$$\begin{aligned}
\hat{f}^{(m+1)}(k) &= \int_a^b e^{-ikx} f^{(m+1)}(x)dx \\
&= [e^{-ikx} f^{(m)}(x)]_a^b + ik \int_a^b e^{-ikx} f^{(m)}(x)dx \\
&= (ik)^{m+1} \hat{f}(x).
\end{aligned}$$

Therefore, we can deduce that $\hat{f}^{(n)}(x) = (ik)^n \hat{f}(x)$ for all n.

Let $\{Y_n\}$ denote a sequence of normal independent and identically distributed random variables that represent the jumps of a randomly selected particle. We define a **random walk** as the random variable, $S_n = Y_1 + \cdots + Y_n$, which represents the position of the particle after n steps. We can reinterpret this random walk by considering the position of a particle at time $t > 0$ and some scaling constant c (i.e. let $n = ct$). Furthermore, suppose that $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n^2] = \sigma^2 > 0$ and let $f(x)$ be the probability density function of Y_n. By considering the Taylor series expansion, we can deduce the following:

$$\begin{aligned}\hat{f}(k) &= \int e^{-ikx} f(x)dx \\ &= \int \left(1 - ikx + \frac{1}{2!}(-ikx)^2 + \cdots\right) f(x)dx \\ &= \int f(x)dx - ik \int xf(x)dx - \frac{k^2}{2}\int x^2 f(x)dx + o(k^2) \\ &= 1 - \frac{\sigma^2 k^2}{2} + o(k^2)\end{aligned}$$

where the integrals are evaluated over the defined range of x. Note that by linearity of expectation, we have,

$$\mathbb{E}[e^{-ikS_n}] = \mathbb{E}[e^{-ik(Y_1+Y_2+\cdots+Y_n)}] = \mathbb{E}[e^{-ikY_1}]\cdots\mathbb{E}[e^{-ikY_n}] = \mathbb{E}[e^{-ikY}]^n = \hat{f}(k)^n.$$

Let $u(x,t)$ be the probability density function of a random particle at time $t > 0$ and distance x. Since u is a probability density function, we require the scaling constant $c^{-1/2}$ to normalize S_n. Therefore,

$$\mathbb{E}[e^{-ikc^{-1/2}S_n}] = \left(1 - \frac{\sigma^2 k^2}{2c} + o(c^{-1})\right)^n.$$

By using $\left(1 + \frac{r}{n} + o(n^{-1})\right)^n \to e^r$, letting $c \to \infty$ and $n \to \infty$ yields,

$$\exp\left(-\frac{1}{2}t\sigma^2 k^2\right) = \mathbb{E}[e^{-ikZ}] = \hat{u}(k,t)$$

where Z is a normalized random variable. It is plain that $\hat{u}(k,t)$ solves the following differential equation:

$$\frac{d\hat{u}(k,t)}{dt} = -\frac{\sigma^2}{2}k^2\hat{u}(k,t) = \frac{\sigma^2}{2}(ik)^2\hat{u}(k,t).$$

Inverting the differential equation yields the traditional diffusion model. Similarly, inverting the solution of the differential equation yields the closed-form solution of the diffusion model. More details on the inversion are supplemented in Appendix D.

Consequentially, note that by Theorem D.1, since

$$\mathbb{E}[e^{-ikc^{-1/2}S_n}] \to \exp\left(-\frac{1}{2}t\sigma^2k^2\right) = \mathbb{E}[e^{-ikZ}] = \int e^{-ikx}\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{1}{2}t\sigma^2k^2\right),$$

we have,

$$n^{-1/2}S_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}} \Rightarrow Z$$

when Z is Brownian motion with mean zero and variance $\sigma^2 t$. The convergence of the distribution of random variables is commonly renowned as the **central limit theorem**.

We will now extend this proof to derive the fractional diffusion model. Let X be a Pareto random variable. Then, for all n, $\mathbb{P}[X \leq x] = 1 - Cx^{-\alpha}$ for constant $C > 0$, $x \geq C^{1/\alpha}$ and $1 < \alpha < 2$. Taking the derivative of both sides yields the probability density function denoted as:

$$f(x) = \begin{cases} C\alpha x^{-\alpha-1} & x \geq C^{1/\alpha} \\ 0 & x < C^{1/\alpha} \end{cases}$$

Let $0 < p < \alpha$. Then the pth moment is given in the following computation:

$$\begin{aligned}\mathbb{E}[X^p] &= \int x^p f(x)dx \\ &= C\alpha\int_{C^{1/\alpha}}^{\infty} x^{p-\alpha-1}dx \\ &= C\alpha\left[\frac{x^{p-\alpha}}{p-\alpha}\right]_{C^{1/\alpha}}^{\infty} = \frac{\alpha}{\alpha-p}C^{p/\alpha}\end{aligned}$$

Given that $1 < \alpha < 2$, the first moment exists and the second moment is undefined. This implies that the mean of X exists and the variance of X doesn't exist, and thus, the central limit theorem does not hold. We will have to take this into consideration as we construct the fractional diffusion model.

We state the following proposition:

Proposition 1.1. *Let X be a Pareto random variable with probability density function defined as above for some $1 < \alpha < 2$. Then, as $k \to 0$, the Fourier transform of $f(x)$ is*

$$\hat{f}(k) = 1 - C^{1/\alpha}\frac{\alpha}{\alpha-1}ik + C\frac{\Gamma(2-\alpha)}{\alpha-1}(ik)^{\alpha} + O(k^2).$$

Proof. Note that,

$$\begin{aligned}
\mathbb{E}[e^{-ikX}] &= \int_{C^{1/\alpha}}^{\infty} e^{-ikx} C\alpha x^{-\alpha-1}dx \\
&= \int_{C^{1/\alpha}}^{\infty} \left[1 - ikx + \left(e^{-ikx} - 1 + ikx\right)\right] C\alpha x^{-\alpha-1}dx \\
&= 1 - C^{1/\alpha}\frac{\alpha}{\alpha-1}ik + \int_0^{\infty} \left(e^{-ikx} - 1 + ikx\right) C\alpha x^{-\alpha-1}dx \\
&\quad - \int_0^{C^{1/\alpha}} \left(e^{-ikx} - 1 + ikx\right) C\alpha x^{-\alpha-1}dx.
\end{aligned}$$

Let

$$J(\alpha) := C\int_0^{\infty} \left(e^{-ikx} - 1 + ikx\right) \alpha x^{-\alpha-1}dx$$

and for $s > 0$, define,

$$J_s(\alpha) := C\int_0^{\infty} \left(e^{(-ik-s)x} - 1 + (ik-s)x\right) \alpha x^{-\alpha-1}dx.$$

By the Dominated Convergence Theorem (Theorem D.3), the boundary conditions vanish. Integration by parts yields the following:

$$\begin{aligned}
J_s(\alpha) &= C(-ik-s)\int_0^{\infty} \left(e^{(-ik-s)x} - 1\right) x^{-\alpha}dx \\
&= C\frac{-ik-s}{\alpha-1}\int_0^{\infty} \left(e^{(-ik-s)x} - 1\right) (\alpha-1)x^{-(\alpha-1)-1}dx.
\end{aligned}$$

Another integration by parts calculation yields,

$$J_s(\alpha) = C\frac{-ik-s}{\alpha-1}\left[\left[\left(e^{(-ik-s)x} - 1\right)(-x^{-(\alpha-1)})\right]_0^{\infty} + (-ik-s)\int_0^{\infty} e^{(-ik-s)x}x^{-(\alpha-1)}dx\right]$$

The boundary terms vanish since $e^{(-ik-s)x} - 1 = O(x)$ as $x \to 0$. Using the characteristic function of a gamma probability density function:

$$\int_0^{\infty} e^{ikx}\frac{b^a}{\Gamma(a)}x^{a-1}e^{-by}dx = \left(1 - \frac{ik}{b}\right)^{-a}$$

we have,

$$J_s(\alpha) = C\frac{-ik-s}{\alpha-1}(-ik-s)\frac{\Gamma(2-\alpha)}{s^{2-\alpha}}\left(1+\frac{ik}{s}\right)^{-(2-\alpha)}$$

or,

$$J_s(\alpha) = C\frac{ik+s}{\alpha-1}\Gamma(2-\alpha)(s+ik)^{\alpha-1} = \frac{C\Gamma(2-\alpha)}{\alpha-1}(s+ik)^{\alpha}.$$

Using Theorem D.3, as we let $s \to 0$, we have,

$$J_s(\alpha) \to J(\alpha) = \frac{C\Gamma(2-\alpha)}{\alpha-1}(ik)^{\alpha}.$$

For the second integral, by using Taylor series expansion, note that for all $x, k \in \mathbb{R}$, we have

$$|e^{-ikx} - 1 + ikx| \leq \frac{(kx)^2}{2!}.$$

Thus,

$$\left|\int_0^{C^{1/\alpha}} \left(e^{-ikx} - 1 + ikx\right) C\alpha x^{-\alpha-1}dx\right| \leq \frac{k^2}{2}\int_0^{C^{1/\alpha}} C\alpha x^{-\alpha-1}dx$$

Or,

$$\left|\int_0^{C^{1/\alpha}} \left(e^{-ikx} - 1 + ikx\right) C\alpha x^{-\alpha-1}dx\right| \leq \frac{k^2}{2}C\alpha\left[\frac{x^{2-\alpha}}{2-\alpha}\right]_0^{C^{1/\alpha}} = \frac{k^2}{2}\frac{\alpha}{2-\alpha}C^{2/\alpha} = O(k^2).$$

Thus,

$$\hat{f}(k) = 1 - C^{1/\alpha}\frac{\alpha}{\alpha-1}ik + C\frac{\Gamma(2-\alpha)}{\alpha-1}(ik)^{\alpha} + O(k^2).$$

□

In the derivation of the fractional diffusion model, we will follow a similar method used to prove the traditional case. Suppose that $\{Y_n\}$ denote a sequence of Pareto random variables with mean zero. These random variables can be constructed by letting Y_n be independent and identically distributed with $X - \mathbb{E}[X]$.

Using Proposition 1.1 for $C = \dfrac{\alpha - 1}{\Gamma(2-\alpha)}$ and the Taylor series expansion for e^z, as $k \to 0$, we have,

$$\begin{aligned}\mathbb{E}[e^{-ik(X-\mathbb{E}[X])}] &= [1 - ik\mathbb{E}[X] + (ik)^\alpha + O(k^2)]\left[1 + ik\mathbb{E}[X] + \frac{1}{2!}(ik\mathbb{E}[X])^2 + O(k^3)\right] \\ &= 1 + (ik)^\alpha + O(k^2).\end{aligned}$$

Let $S_n = Y_1 + \cdots + Y_n$ be a random walk denoting the position of the particle after n steps and let $f(x)$ be the probability density function of Y_n. Recall that n can be reinterpreted by considering the position of a particle at time $t > 0$ and some scaling constant c (i.e. let $n = ct$). Let $u(x,t)$ be the probability density function of a random particle at time $t > 0$ and distance x. Since u is a probability density function, we require the scaling constant $c^{-1/\alpha}$ to normalize the sum S_n. Therefore,

$$\mathbb{E}[e^{-ikc^{-1/\alpha}S_n}] = \left(1 - \frac{(ik)^\alpha}{c} + O(c^{-2/\alpha})\right)^n.$$

Letting $c \to \infty$ and $n \to \infty$ yields,

$$e^{t(ik)^\alpha} = \mathbb{E}[e^{-ikZ}] = \hat{u}(k,t)$$

where Z is a normalized random variable. Again, it is clear that $\hat{u}(k,t)$ solves the following differential equation:

$$\frac{d\hat{u}(k,t)}{dt} = (ik)^\alpha \hat{u}(k,t).$$

We refer to Example 1.1 to construct a definition for the fractional derivative. For the purpose of consistency, we define the **fractional derivative** $\dfrac{d^\alpha f(x)}{dx^\alpha}$ to be a function whose Fourier transform is $(ik)^\alpha \hat{f}(k)$. Inverting the above differential equation yields the fractional diffusion model provided below.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$$

Specific details on the inversion are supplemented in Appendix D.

Consequentially, note that by Theorem D.1, since

$$\mathbb{E}[e^{-ikc^{-1/\alpha}S_n}] \to e^{t(ik)^\alpha} = \mathbb{E}[e^{-ikZ}] = \int e^{-ikx}u(x,t)dx$$

we have,

$$c^{-1/\alpha}S_n = \frac{Y_1 + \cdots + Y_n}{n^{1/\alpha}} \Rightarrow Z$$

when Z is $\alpha-$stable motion with mean zero. The convergence of the distribution of random variables is renowned as the **extended central limit theorem**.

The above fractional diffusion equation models super-diffusion. In the traditional case, the diffusion equation models the concentration of particles from one unit to another. Particles contained in a unit of higher concentration have the tendency to disperse into adjacent units of lower concentration. This physical behavior is known as standard diffusion.

However, the fractional diffusion equation models the concentration of particles from one unit to many adjacent units. The concentration of particles is spread partially among all adjacent units of lower concentration. The amount of particles entering into these units are distributed exponentially. Specifically, if the unit is closer to the source, then there is an increased amount of particles that will enter the unit. This distribution of particles is known as super-diffusion.

Compared to traditional diffusion, super-diffusion is a generalized measure of gradient change. This measure is also more realistic in many scientific applications. For these purposes, we tackle the inconsistencies of the Black-Scholes model by considering its fractional counterpart.

The solution of the fractional diffusion model is positively skewed with a heavy power-law tail. Specifically, as $x \to \infty$, then for some constant $A = A(C, t, \alpha) > 0$, we have $u(x,t) = Ax^{-\alpha-1} + o(x^{-\alpha-1})$. [7] Note that the tail behavior does not disappear as $u(x,t)$ is passed to limits. This is similar to how implied volatility behaves under changes of maturity dates for S&P 500 options pricing after the stock market crash of 1987. As discussed previously, as maturity increases, there exists leptokurtosis in the return distribution and therefore the tails of this distribution appears fat compared to the tails of the Gaussian.

Recall that in the necessary assumptions of the Black-Scholes model, we require a generalization of Brownian motion to compensate for the recent behavior of option prices. We consider a specific class of stochastic processes called Levy $\alpha-$stable motion and readjust our assumptions so that asset prices satisfy a Geometric Levy motion. Given that the parameter $\alpha \in (1, 2]$, fractional derivatives will arise and become crucial in the development of the Finite Moment Log Stable (FMLS) model. This will require some background information on $\alpha-$stable motion and fractional derivatives. In the next section, we will first discuss about fractional derivatives and its interesting properties.

1.4 Fractional Derivatives

As mentioned in the previous section, we have defined the fractional derivative as a function $\frac{d^\alpha f(x)}{dx^\alpha}$ whose Fourier transform is $(ik)^\alpha \hat{f}(k)$. We will now provide an alternate definition of fractional derivatives using standard calculus intuition.

Recall that the first derivative is defined by the following limit:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}.$$

We will assume that this limit exists. For higher order derivatives, we have,

$$\frac{d^n f(x)}{dx^n} = \lim_{h \to 0} \frac{\Delta^n f(x)}{h^n}.$$

where $\Delta^n f(x) = \sum_{j=0}^{\infty} \binom{n}{j} (-1)^j f(x - jh)$ for $n \in \mathbb{N}$. Extending this definition of the difference operator for $\alpha \in \mathbb{R}$ yields the following:

$$\frac{d^\alpha f(x)}{dx^\alpha} = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha} = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh).$$

This is known as the **Grunwald-Letnikov finite difference** form for the fractional derivative. We use the convention that

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha-j+1)}.$$

We will now state a proposition that shows that the two definitions of the fractional derivative agree with each other.

Proposition 1.2. *Let $f(x)$ be a bounded function such that f and its derivatives up to order $n > 1+\alpha$ exist and are absolutely integrable for all $\alpha \in \mathbb{R}$. Then, the Grunwald-Letnikov fractional derivative exists and as $h \to 0$, the Fourier transform of $\frac{\Delta^\alpha f(x)}{h^\alpha}$ approaches $(ik)^\alpha \hat{f}(k)$.*

Proof. Let $w_j := \binom{\alpha}{j}(-1)^j$. Then, by the Binomial formula, since,

$$\sum_{j=0}^{\infty} w_j = \sum_{j=0}^{\infty} \binom{\alpha}{j}(-1)^j = (1+(-1))^\alpha = 0$$

we have,

$$\sum_{j=0}^{\infty} |w_j| = \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}(-1)^j\right| < \infty.$$

With the assumption that f is bounded, then for $-\infty < x < \infty$, we have the following uniform convergence:

$$\Delta^\alpha f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j}(-1)^j f(x-jh) < \infty.$$

And therefore the Grunwald-Letnikov fractional derivative exists.

We will now propose a lemma that will assist us in proving the second half of the theorem:

Lemma 1.2. *If $f(x)$ and all of its derivatives up to order n exist and are absolutely integrable, then for all $k \in \mathbb{R}$, there exists constant $C > 0$ such that,*

$$|\hat{f}(k)| \leq \frac{C}{1+|k|^n}.$$

Proof. Note that for $|k| < 1$, since $1 + |k|^n \leq 2$,

$$(1 + |k|^n)|\hat{f}(k)| \leq 2\left|\int e^{-ikx} f(x)dx\right| \leq 2\int |f(x)|dx.$$

Therefore,

$$|\hat{f}(k)| \leq \frac{2}{1 + |k|^n}\int |f(x)|dx.$$

Alternatively, note that by Example 1.1, we have,

$$\hat{f}(k) = (ik)^{-n}\int e^{-ikx} f^{(n)}(x)dx.$$

For $|k| \geq 1$, since $1 + |k|^n \leq 2|k|^n$,

$$(1 + |k|^n)|\hat{f}(k)| \leq 2|k|^n \left|(ik)^{-n}\int e^{-ikx} f^{(n)}(x)dx\right| \leq 2\int |f^{(n)}(x)|dx.$$

Therefore,

$$|\hat{f}(k)| \leq \frac{2}{1 + |k|^n}\int |f^{(n)}(x)|dx.$$

Letting $C := \max\left\{2\int |f(x)|dx\ ,\ 2\int |f^{(n)}(x)|dx\right\}$ proves the lemma. □

Using Lemma 1.2, it follows that for all k,

$$|(ik)^\alpha \hat{f}(k)| \leq \frac{C|k|^\alpha}{1 + |k|^n}.$$

Given that $n > 1 + \alpha$, $(ik)^\alpha \hat{f}(k)$ is absolutely integrable. By applying Theorem D.2, we can deduce that there exists a function with Fourier transform $(ik)^\alpha \hat{f}(k)$. By an earlier discussion, we have denoted this function as the fractional derivative.

Since,

$$\int e^{-ikx} f(x - a)dx = \int e^{-ik(y+a)} f(y)dy = e^{-ika}\int e^{-iky} f(y)dy == e^{-ika}\hat{f}(k),$$

the Fourier Transform of $\Delta^n f(x)$ is:

$$\begin{aligned}\int e^{-ikx} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x-jh)dx &= \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j \int e^{-ikx} f(x-jh)dx \\ &= \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-ikjh} \hat{f}(k) \\ &= (1-e^{-ikh})^{\alpha} \hat{f}(k)\end{aligned}$$

And the Fourier Transform of $h^{-\alpha}\Delta^n f(x)$ is:

$$\begin{aligned}h^{-\alpha}(1-e^{-ikh})^{\alpha}\hat{f}(k) &= h^{-\alpha}(ikh)^{\alpha}\left(\frac{1-e^{-ikh}}{ikh}\right)^{\alpha}\hat{f}(k) \\ &= (ik)^{\alpha}\left(1+O(ihk)\right)^{\alpha}\hat{f}(k).\end{aligned}$$

Letting $h \to 0$ yields that for all h,

$$\int e^{-ikx} h^{-\alpha} \Delta^n f(x) dx \to (ik)^{\alpha} \hat{f}(k).$$

Since the Fourier transform of $\dfrac{\Delta^n f(x)}{h^{\alpha}}$ converges pointwise to $\dfrac{d^{\alpha} f(x)}{dx^{\alpha}}$, the continuity theorem for Fourier transforms proves the theorem. □

Remark: The term w_j defined in the previous proof is known as a **Grunwald weight**. Note that by definition and properties of $\Gamma(\cdot)$,

$$w_j = \binom{\alpha}{j}(-1)^j = \frac{(-1)^j\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} = \frac{-\alpha\Gamma(j-\alpha)}{\Gamma(j+1)\Gamma(1-\alpha)}.$$

Applying Stirling's formula $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ as $x \to \infty$ yields,

$$w_j \sim \frac{-\alpha}{\Gamma(1-\alpha)}\sqrt{\frac{j-\alpha-1}{j}}\left(\frac{j-\alpha-1}{j}\right)^{j-\alpha-1} j^{-\alpha-1}e^{\alpha+1}.$$

Letting $j \to \infty$ yields,

$$w_j \sim \frac{-\alpha}{\Gamma(1-\alpha)} j^{-\alpha-1}.$$

Note that,

$$\frac{\Delta^{\alpha} f(x)}{\Delta x^{\alpha}} = (\Delta x)^{-\alpha}\left[f(x) + \sum_{j=1}^{\infty} w_j f(x-j\Delta x)\right].$$

Let $0 < \alpha < 1$ and for $j \geq 1$, define $b_j = -w_j$ so that,

$$b_j \sim \frac{\alpha}{\Gamma(1-\alpha)} j^{-\alpha-1} \text{ as } j \to \infty, \text{ and } \sum_{j=1}^{\infty} b_j = 1.$$

Then,

$$\begin{aligned}
\frac{\Delta^\alpha f(x)}{\Delta x^\alpha} &= (\Delta x)^{-\alpha} \left[f(x) \sum_{j=1}^{\infty} b_j + \sum_{j=1}^{\infty} (-b_j) f(x - j\Delta x) \right] \\
&= (\Delta x)^{-\alpha} \sum_{j=1}^{\infty} [f(x) - f(x - j\Delta x)] b_j \\
&\approx \sum_{j=1}^{\infty} [f(x) - f(x - j\Delta x)] \frac{\alpha}{\Gamma(1-\alpha)} (j\Delta x)^{-\alpha-1} \Delta x \\
&\approx \int_0^{\infty} [f(x) - f(x-y)] \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy.
\end{aligned}$$

Passing to limits yields,

$$\frac{d^\alpha f(x)}{dx^\alpha} = \int_0^{\infty} [f(x) - f(x-y)] \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy.$$

After applying integration by parts, we have

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{d}{dx} f(x-y) y^{-\alpha} dy.$$

Rename $u = x - y$ and taking the derivative outside yields,

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} f(u)(x-u)^{-\alpha} du.$$

The above form is renowned as the **Riemann-Liouville fractional derivative** for $0 < \alpha < 1$. Analogous forms can be constructed for other values of α. This definition is necessary for the derivation of the Fractional Black-Scholes equation satisfying the FMLS model. We will eventually restrict our attention for all α satisfying $1 < \alpha < 2$.

In this section, we will develop a framework for $\alpha-$stable random variables and discuss about some properties they hold. We will further use these properties to construct the Finite Moment Log Stable model. Readers can refer to Samorodinitsky and Taqqu for proof of these properties. [12]

Firstly, we propose the definition of a Levy process to construct a similar definition for $\alpha-$stable motion.

Definition 1.2. *A time-dependent random variable X is said to be a* ***Levy process*** *if and only if X has independent and stationary increments. The* ***Levy-Khintchine representation*** *of the log-characteristic function of X is provided below:*

$$\log \mathbb{E}[e^{ikX(t)}] = mitk - \frac{1}{2}\sigma^2 tk^2 + t\int_{\mathbb{R}\setminus\{0\}} (e^{ikx} - 1 - ikxI_{|x|<1})dW$$

with drift $m \in \mathbb{R}$, volatility $\sigma \geq 0$, indicator function I and Levy measure W satisfying:

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, x^2\}dW < \infty.$$

If Levy measure dW is of the form $w(x)dx$, we define $w(x)$ to be the Levy density. For our purposes, the Levy density of $\alpha-$stable process is defined as:

$$w_{LS}(x) = \begin{cases} \frac{1}{2}(1-\beta)D|x|^{-1-\alpha} & x < 0 \\ \frac{1}{2}(1+\beta)Dx^{-1-\alpha} & x > 0 \end{cases}$$

where $D > 0$, and skewness $-1 < \beta < 1$, and $0 < \alpha \leq 2$. Applying Definition 1.2 for this Levy density yields the following characteristic exponent $\Psi(k)$:

$$\Psi(k) = ik\mu - \frac{1}{2}|k|^\alpha \sigma^\alpha \left(1 - i\beta(\text{sgn}(k)) \tan \frac{\pi\alpha}{2}\right)$$

with drift μ and volatility σ. This motivates us to study a class of Levy processes called $\alpha-$stable processes. We will shortly show that this class of stochastic processes satisfy the desired conditions reflected in the S&P options market. We provide the following definition:

Definition 1.3. *A random variable X is said to have an $\alpha-$stable distribution if for the given parameters, we have the following conditions: index of stability $1 < \alpha \leq 2$, drift μ, dispersion $\sigma \geq 0$, and skewness parameter $-1 \leq \beta \leq 1$, then the characteristic function of X, $\varphi_X = \varphi_X(k)$, is:*

$$\varphi_X = \mathbb{E}[e^{ikX}] = \exp\left[ik\mu - |k|^\alpha \sigma^\alpha \left(1 - i\beta(sgn(k)) \tan\frac{\pi\alpha}{2}\right)\right]$$

Letting $\alpha = 2$ yields the characteristic function of a Gaussian random variable W with mean μ and variance $2\sigma^2$. Specifically, we have,

$$\varphi_W = \mathbb{E}[e^{ikW}] = \exp\left[ik\mu - k^2\sigma^2\right].$$

In this class of stochastic processes, the thickness of the tails remains invariant under time. This is shown with the following property.

Property 1.1. *Let X be $\alpha-$stable with $\alpha < 2$, dispersion σ, skewness β, and drift μ. The tails remain "fat" and the tail probabilities are given as follows:*

$$\lim_{\lambda\to\infty} \lambda^\alpha P(\pm X > \lambda) = C(\alpha)\frac{1 \pm \beta}{2}\sigma^\alpha$$

where,

$$C(\alpha) = \left(\int_0^\infty \frac{\sin x}{x^\alpha} dx\right)^{-1} = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \alpha \neq 1 \\ 2/\pi & \alpha = 1 \end{cases}.$$

When X is maximally negatively skewed with zero drift (i.e. $\beta = -1$, $\mu = 0$), we have

$$P(X > \lambda) \sim \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}}\left(\frac{\lambda}{\alpha C(\sigma,\alpha)}\right)^{-\alpha/(2\alpha-2)} \exp\left(-(\alpha-1)\left(\frac{\lambda}{\alpha C(\sigma,\alpha)}\right)^{\alpha/(\alpha-1)}\right)$$

where,

$$C(\sigma,\alpha) = \sigma\left(\cos\frac{(2-\alpha)\pi}{2}\right)^{-1/\alpha}$$

This property also suggests that for $\alpha < 2$, the tail probabilities behave like $\lambda^{-\alpha}$ and therefore exhibits a "power-law"-like behavior. Note that since the tail behavior does not flatten out as maturity increases, $\alpha-$stable variables violate the implications of the central limit theorem. This suggests that variance for log return is infinite. The questions of the existence of a martingale measure and whether option values are finite or infinite arise.

By setting the condition of $\alpha-$stable motion to have maximum negative skewness, we claim that conditional moments of all orders exist for $1 < \alpha < 2$. To show this, we need to propose three additional properties of $\alpha-$stable motion that will assist us in the proof of our claim. We will first start off by providing a definition and some notation.

Definition 1.4. *Let Y be a random variable with probability density function $f(x)$. We define the **Laplace transform** of $f(x)$ to be the following:*

$$\tilde{f}(s) = \mathcal{B}\{f(t)\} := \mathbb{E}[e^{-sX}] = \int_0^\infty e^{-st}f(t)dt.$$

*We define the **two-sided Laplace transform** of $f(x)$ to be the following:*

$$\tilde{f}(s) = \mathcal{B}\{f(t)\} := \mathbb{E}[e^{-sX}] = \int_{-\infty}^\infty e^{-st}f(t)dt.$$

Similarly to how we have notated $W_t = W(t)$ for Brownian motion, we also propose a notation for Levy $\alpha-$stable motion. Formally, we will denote $L_t^{\alpha,\beta} = L^{\alpha,\beta}(t)$ as the standardized Levy $\alpha-$stable motion with tail index $0 < \alpha \leq 2$, and skew parameter $-1 \leq \beta \leq 1$. Furthermore, by Definition 1.3, given that there are four parameters, we will conveniently notate the distribution of $\alpha-$stable motion as $L_\alpha(\mu,\sigma,\beta)$. We say that if X has a stable distribution with the above parameters, we notate this as $X \sim L_\alpha(\mu,\sigma,\beta)$.

We will now proceed to mention some additional properties:

Property 1.2. *Let $X \sim L_\alpha(\mu, \sigma, \beta)$. The two-sided Laplace transform of X is not finite unless $\beta = 1$. For $\beta = 1$, we have the following Laplace transform:*

$$\mathbb{E}[e^{-sX}] = \exp\left(-s\mu - s^\alpha \sigma^\alpha \sec \frac{\alpha\pi}{2}\right).$$

Property 1.3. *For any $0 < \alpha < 2$,*

$$X \sim L_\alpha(0, \sigma, \beta) \Leftrightarrow -X \sim L_\alpha(0, \sigma, -\beta).$$

Property 1.4. *For any $0 < \alpha < 2$, we have $\mathbb{E}[|X|^p] < \infty$ for any $0 < p < \alpha$ and $\mathbb{E}[|X|^p] = \infty$ for $p > \alpha$.*

What makes $\alpha-$stable processes so appealing for the construction of our new model is described by these properties. As mentioned previously, Property 1.1 suggests that the tail behavior is invariant to maturity. We say that this property exhibits **self-similarity**. Property 1.2 defines the Laplace transform for $\alpha-$stable random variable X with maximum positive skewness, and Property 1.3 allows us to determine a closed form for the Laplace Transform by relating X to $-X$. Lastly, Property 1.4 describes the finiteness condition for the moments of X depending on α.

Let $L_t^{\alpha,\beta}$ be a Levy $\alpha-$stable process with skew parameter β. Define S_t to be the stock price and assume that S_t satisfies the following stochastic differential equation: for time $0 < t < T$, index of stability $1 < \alpha < 2$, and volatility $\sigma > 0$,

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dL_t^{\alpha,-1}$$

where r and q respectively denote deterministic parameters corresponding to the risk free rate and dividend yield. We selectively restrict $\beta = -1$ to obtain finite moments of S_t and negative skewness in the return density distribution. Specifically by Properties 1.3 and 1.2, for $n > 0$,

$$\mathbb{E}[\exp\left(n\sigma L_t^{\alpha,-1}\right)] = \exp\left(-tn^\alpha \sigma^\alpha \sec \frac{\pi\alpha}{2}\right) < \infty.$$

This suggests that a martingale measure exists for $\beta = -1$. Furthermore, we restrict α to satisfy $1 < \alpha < 2$ so that S_t remains unbounded. We define the above stochastic differential equation to be the **Finite Moment Log Stable Model** (FMLS model). In regards to this model, we state and prove the following proposition. The proposition is two-fold.

Proposition 1.3. *Let $s_\tau := \log \frac{S_T}{S_t}$ be the log return over maturity $\tau = T - t$. Then,*

i) $s_\tau \sim L_\alpha((r - q + \mu)\tau, \sigma\tau^{1/\alpha}, -1)$ with convexity adjustment $\mu = \sigma^\alpha \sec \frac{\pi\alpha}{2}$.

ii) For all $n \geq 0$, the nth conditional moment of S_T is well defined and given as,

$$\mathbb{E}[S_T^n] = S_T^n \exp\left(n(r - q + \mu)\tau - \tau(n\sigma)^\alpha \sec \frac{\pi\alpha}{2}\right) < \infty$$

Proof. Let S_t satisfy the above stochastic differential equation for all $0 < t \leq T$, $1 < \alpha < 2$, and $\sigma > 0$. We can re-express S_T in the following exponential form:

$$S_T = S_t e^{(r-q)\tau} \exp\left(\mu\tau + \sigma L_\tau^{\alpha,-1}\right)$$

where μ is specifically chosen so that $\mathbb{E}[\left(\mu\tau + \sigma L_\tau^{\alpha,-1}\right)] = 1$. Note that by Property 1.3 and 1.2, we have

$$\mathbb{E}[\exp(\sigma L_\tau^{\alpha,-1})] = \mathbb{E}[\exp(-\sigma L_\tau^{\alpha,1})] = \exp\left(-\tau\sigma^\alpha \sec \frac{\pi\alpha}{2}\right).$$

This requires $\mu = \sigma^\alpha \sec \frac{\pi\alpha}{2} < \infty$. With this choice of μ, we have the following exponential form,

$$S_T = S_t e^{(r-q+\mu)\tau} \exp\left(\sigma L_\tau^{\alpha,-1}\right).$$

Therefore, the log return s_τ satisfies

$$s_\tau = \log \frac{S_T}{S_t} = (r + q + \mu)\tau + \sigma L_\tau^{\alpha,-1}.$$

Note that s_τ is $\alpha-$stable distributed with mean $(r-q+\mu)\tau$, dispersion $\sigma\tau^{1/\alpha}$, and skewness $\beta = -1$. We can further deduce that by Property 1.4, the variance of s_τ or any moments of order higher than α is not finite. This proves the first statement of the proposition.

Note that for $n \geq 0$, by Property 1.2,

$$\begin{aligned}\mathbb{E}[S_T^n] &= S_t^n e^{n(r-q+\mu)\tau}\mathbb{E}\left[\exp\left(n\sigma L_\tau^{\alpha,-1}\right)\right] \\ &= S_t^n \exp\left(n(r-q+\mu)\tau - \tau(n\sigma)^\alpha \sec\frac{\pi\alpha}{2}\right) < \infty.\end{aligned}$$

This proves the second statement of the proposition. The proof is complete. □

With the proposition proven, the FMLS model performs better than the standard Black-Scholes model in reflecting how S&P option prices behave after the 1987 U.S. stock market crash. As desired, under the FMLS model, the return distribution contains "fat" tails with the extra condition of infinite return moments and finite price moments.

1.6 Derivation of Black-Scholes Equation Under FMLS Model

In Section 1.2, we have derived the Black-Scholes model and its corresponding partial differential equation using the assumption that asset prices satisfy a geometric Brownian motion. Alternatively, by applying Ito's Chain Rule (Theorem B.1), the Black-Scholes model states that for asset price S_t, the log price satisfies the following stochastic differential equation:

$$d(\log S_t) = \left(r - q - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

with risk-free rate $r > 0$, dividend yield $q > 0$ and volatility $\sigma \geq 0$. The price of a European call option $C(S, t)$ with S_t satisfying the above stochastic differential equation is given in Section 1.2. With a change of variables $x_t = \log S_t$, the Black-Scholes equation can be rewritten as the following advection-diffusion type equation:

$$\frac{\partial C(x,t)}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial C(x,t)}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial C(x,t)}{\partial x} = rC(x,t)$$

with $C(x,t)$ representing the price of a European call option under the Black-Scholes model and risk-free rate r.

In this section, we generalize the derivation of the partial fractional differential equation that arise from the FMLS model. More details can be referred in Cartea and del-Castillo-Negrete. [4] We readjust the Black-Scholes model by assuming that the stock price follows a geometric Levy process with maximum negative skewness. This motivates us to consider the following stochastic differential:

$$d(\log S_t) = (r - q - v)\,dt + \sigma dL_t^{\alpha,-1}.$$

The above equation has the following solution:

$$S_T = S_t \exp\left((r - q - v)\tau + \sigma \int_t^T dL_\tau^{\alpha,-1}\right)$$

where $\tau = T - t$ and v is a convexity adjustment so that $\mathbb{E}[S_T] = e^{(r-q)\tau} S_t$. In determining the corresponding fractional partial differential equation that models the behavior of the value of a European call option $V(x,t)$, we prove the following proposition that provides a differential equation which governs the behavior of its Fourier transform.

Proposition 1.4. *Let $V(x,t)$ be the value of a European call option under the convention that $x := x(t) = \log S_t$, for asset price S_t with $0 \leq t \leq T$. We assume that S_t satisfies a Geometric Levy distribution. Then, the Fourier transform of $V(x,t)$, $\hat{V}(x,t)$, satisfies the following differential equation:*

$$\frac{\partial \hat{V}(k,t)}{\partial t} = (r + ik(r - v) - \Psi(-k))\,\hat{V}(k,t)$$

where $\Psi(k)$ is the characteristic exponent for some Levy process.

Remark: For the FMLS process, by Definition 1.2, we have,

$$\Psi(k) = ik\mu - \frac{1}{2}|k|^\alpha \sigma^\alpha \left(1 - i\beta(\text{sgn}(k)) \tan \frac{\pi\alpha}{2}\right).$$

Equivalently, we have,

$$\Psi(k) = -\frac{1}{4}\sigma^\alpha \sec\frac{\alpha\pi}{2}\left((1-\beta)(ik)^\alpha + (1+\beta)(-ik)^\alpha\right).$$

Thus, for $\beta = -1$, we have,

$$\Psi(-k) = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}(-ik)^\alpha.$$

More details are provided in Benson, Meerschaert, and Wheatcraft. [1]

Proof. Assuming the market is risk-free, the value of the European call option can be written in terms of the expected value of the final payoff. Let $\Pi(x, T)$ be the final payoff of the portfolio. Then for interest rate r and maturity $\tau := \tau(t) = T - t$, we have,

$$dV(x_t, t) = r\mathbb{E}[\Pi(x_T, T)]dt = -r\mathbb{E}[\Pi(x_T, T)]d\tau.$$

Equivalently,

$$V(x_t, t) = e^{-r(T-t)}\mathbb{E}[\Pi(x_T, T)].$$

Assuming that $\Pi(x, T)$ has a Fourier transform, applying Theorem D.2 on $\Pi(x_T, T)$ yields,

$$V(x_t, t) = \frac{e^{-r(T-t)}}{2\pi}\mathbb{E}\left[\int_{-\infty}^{\infty} e^{-ikx_T}\hat{\Pi}(k, T)dk\right].$$

By linearity of expectation and using the characteristic function of $x_T = \log S_T$, we have,

$$\begin{aligned} V(x, t) &= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}\left(e^{-ikx_T}\right)\hat{\Pi}(k, T)dk \\ &= \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} e^{-ikx_t - ik(r-v)(T-t)}e^{(T-t)\Psi(-k)}\hat{\Pi}(k, T)dk. \end{aligned}$$

Applying Theorem D.2 on $V(x, t)$ and equating forms yields,

$$\hat{V}(k, t) = e^{[-r-ik(r-v)+\Psi(-k)](T-t)}\hat{\Pi}(k, T)$$

with $\hat{V}(k,T) = \hat{\Pi}(k,T)$. It can easily be checked that the above function solves the differential equation:

$$\frac{\partial \hat{V}(k,t)}{\partial t} = (r + ik(r - v) - \Psi(-k))\,\hat{V}(k,t).$$

□

By the previous remark, we have established that under the FMLS model with maximum negative skewness,

$$\Psi(-k) = -\frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}(-ik)^{\alpha}.$$

With convexity adjustment $v = -\frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}$ in Proposition 1.3, by Proposition 1.4, we have,

$$\frac{\partial \hat{V}(k,t)}{\partial t} = \left(r + ik\left(r + \frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}\right) + \frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}(-ik)^{\alpha}\right)\hat{V}(k,t).$$

Using Theorem D.2 on both sides yields,

$$\frac{\partial V(x,t)}{\partial t} + \left(r + \frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}\right)\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}\frac{\partial^{\alpha}V(x,t)}{\partial x^{\alpha}} = rV(x,t)$$

where,

$$\frac{\partial^{\alpha}V(x,t)}{\partial x^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x} V(u,t)(x-u)^{-\alpha}du.$$

The above fractional partial differential equation is renowned as the Black-Scholes Equation under the FMLS model. This is the equation that we will be studying the solutions for throughout this paper.

Remark: Note that if $\alpha = 2$, the derived Black-Scholes equation degenerates into the classical Black-Scholes equation in its advection-dispersion form, given below:

$$\frac{\partial C(x,t)}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 C(x,t)}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial C(x,t)}{\partial x} = rC(x,t).$$

Chapter 2

Analytic Solution

From the previous chapter, we have established that under the FMLS model, the log price under the risk-neutral measure satisfies the following stochastic differential equation:

$$d(\log S_t) = (r - q - v)dt + \sigma dL_t^{\alpha,-1}$$

where r and q are the risk-free interest rate and the dividend yield respectively, and $v = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}$ represents the convexity adjustment. Let $V(x,t)$ be the price of the European call option with $x = x_t := \log S_t$. From the previous chapter, we have shown that $V(x,t)$ satisfies the Black-Scholes equation under FMLS given below: for all $0 \leq t \leq T$, and $1 < \alpha \leq 2$,

$$\frac{\partial V(x,t)}{\partial t} + \left(r + \frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}\right)\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}\frac{\partial^\alpha V(x,t)}{\partial x^\alpha} = rV(x,t)$$

with boundary conditions,

$$V(x,T) = \Pi(x) := \begin{cases} \max\{e^x - K, 0\} & \text{for European call option} \\ \max\{K - e^x, 0\} & \text{for European put option} \end{cases}$$

where K is the strike price and,

$$\frac{\partial^\alpha V(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x} V(u,t)(x-u)^{-\alpha}du.$$

In this chapter, we will determine a closed-form analytical solution of the Black-Scholes equation using Fourier transforms. We will state a few theorems relating our solution to the Black-Scholes pricing formula provided in the previous chapter. More information can be referred to Chen, Xu, and Zhu. [5]

2.1 Preliminaries

In our derivation of the generalization of the Black-Scholes formula, we require additional definitions and properties of $\alpha-$stable motion. We also require a lemma that will help us prove a theorem discussed later in this thesis.

By Definition 1.3, the characteristic function of $\alpha-$stable random variable X with drift μ, dispersion $\sigma \geq 0$, and skewness parameter $\beta \in [-1, 1]$ is

$$\varphi(k) = \exp\left(ik\mu - |k|^{\alpha}\sigma^{\alpha}\left(1 - i\beta \mathrm{sgn}(k) \tan\frac{\alpha\pi}{2}\right)\right).$$

Letting $\beta = 0$ yields a symmetric distribution with translation constant μ and scaling factor σ^{α}. This can be shown by considering the Levy density function. Namely, for $\beta = 0$ and for some constant $C > 0$, the Levy density becomes

$$w_{LS}(x) = \begin{cases} C|x|^{-1-\alpha} & x < 0 \\ Cx^{-1-\alpha} & x > 0 \end{cases}$$

Note that the Levy density function under this choice of β is symmetrical. Since w_{LS} correlates to the behavior of the distribution, it implies that the Levy distribution is centered and symmetrical. Therefore, for $1 < \alpha \leq 2$, $|\varphi(k)| = \exp(-|k|^{\alpha})$.

In general, under new centring constant β satisfying

$$|\beta| \leq \begin{cases} \alpha & 0 < \alpha < 1 \\ 2 - \alpha & 1 < \alpha < 2 \end{cases}$$

and by removing the drift and dispersion by letting $\mu = 0$ and $\sigma = 1$, the log characteristic function of X, $\Psi(k)$, is defined as follows:

$$\Psi(k) = -|k|^{\alpha} \exp\left(i\frac{\pi\beta}{2}\mathrm{sgn}(k)\right).$$

This motivates the following property.

Property 2.1. *Let $f_{\alpha,\beta}(x)$ be the probability density function of $\alpha-$stable variable X. $f_{\alpha,\beta}$ is the Fourier transform of characteristic function $\varphi(z)$, given as follows*

$$f_{\alpha,\beta}(x) = \frac{1}{\pi} Re \int_0^{\infty} \exp\left(-ixz - z^{\alpha} \exp\left(\frac{i\pi\beta}{2}\right)\right) dz.$$

Now, we propose the definition of a function that plays a central role in the determination of solutions for fractional partial differential equations.

Definition 2.1. *Let $A_i, B_i > 0$, $0 \leq m \leq q$ and $0 \leq n \leq p$ be constants. Let a_j, b_j be complex numbers such that no pole of $\Gamma(b_j - B_j s)$ for $j = 1, \cdots, m$ coincides with any pole of $\Gamma(1 - a_j + A_j s)$ for $j = 1, \cdots, n$. We define the **Fox H-function** to be the following function:*

$$H_{p,q}^{m,n}(z) := H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1, A_1) & \cdots & (a_p, A_p) \\ (b_1, B_1) & \cdots & (b_p, B_p) \end{matrix} \right. \right]$$
$$= \frac{1}{2\pi i} \int_C \frac{\Pi_{j=1}^{m} \Gamma(b_j - B_i s) \Pi_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\Pi_{j=n+1}^{p} \Gamma(a_j + A_j s) \Pi_{j=m+1}^{q} \Gamma(1 - b_j + B_j s)} z^s ds$$

where C is a contour in the complex plane such that $\frac{b_j + k}{B_j}$ and $\frac{a_j - 1 - k}{A_j}$ lie to the right and left of C, respectively.

The probability density function of stable variables can be rewritten in terms of this special function. We will propose the analytic form of the probability density function using the Fox H-function.

Property 2.2. *For $\alpha > 1$, the analytic form of $f_{\alpha,\beta}$ is given as follows:*

$$f_{\alpha,\beta}(x) = \frac{1}{\alpha} H_{2,2}^{1,1}\left[x \left| \begin{matrix} (1 - \frac{1}{\alpha}, \frac{1}{\alpha}) & (1 - \frac{\alpha-\beta}{2\alpha}, \frac{\alpha-\beta}{2\alpha}) \\ (0, 1) & (1 - \frac{\alpha-\beta}{2\alpha}, \frac{\alpha-\beta}{2\alpha}) \end{matrix} \right. \right]$$

Soon, we will see that the closed-form solution of the Black-Scholes equation can also be written in terms of the Fox H-function. Crucial to the understanding of Fox-H functions, we will propose the definition of Mellin transforms:

Definition 2.2. *We define the **Mellin transform** of a function f to be the following:*

$$\mathcal{M}\{f(x)\} = \int_0^\infty x^{s-1} f(x) dx.$$

The inverse Mellin transform is defined as:

$$\mathcal{M}^{-1}\{f(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} f(s) ds$$

where the line integral is taken along a vertical line in the complex plane.

Lastly, we propose the following lemma that emphasizes the properties of the Gamma Function:

Lemma 2.1. *(Gauss Multiplication Formula) For all* $z \in \left\{-\frac{m}{n} : m \notin \mathbb{N}\right\}$,

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - nz} \Gamma(nz).$$

Proof. The Euler form of the Gamma Function is provided below: For all $z \in \mathbb{R} \setminus \mathbb{Z}^-$,

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} = \lim_{m \to \infty} \frac{m^z m!}{z(z+1)(z+2)\dots(z+m)}.$$

Using this definition of the Euler form, Stirling's formula and properties of the Gamma function, namely $\Gamma(z+1) = z\Gamma(z)$, we have:

$$\begin{aligned}
\Gamma\left(z + \frac{k}{n}\right) &= \left(z + \frac{k}{n} - 1\right) \Gamma\left(z + \frac{k}{n} - 1\right) \\
&= \lim_{m \to \infty} \frac{m! m^{z+k/n-1}}{\left(z + \frac{k}{n}\right)\left(z + \frac{k}{n} + 1\right) \cdots \left(z + \frac{k}{n} - 1 + m\right)} \\
&= \lim_{m \to \infty} \frac{\sqrt{2\pi m} \left(\frac{m}{e}\right)^m m^{z+k/n-1}}{\left(z + \frac{k}{n}\right)\left(z + \frac{k}{n} + 1\right) \cdots \left(z + \frac{k}{n} - 1 + m\right)} \\
&= \lim_{m \to \infty} \frac{\sqrt{2\pi} \left(\frac{mn}{e}\right)^m m^{z+k/n-1/2}}{(nz+k)(nz+k+n) \cdots (nz+k-n+mn)}.
\end{aligned}$$

Therefore, making a substitution of $mn \mapsto m$ and reapplying the Euler form of the Gamma Function, Stirling's formula, and properties of the Gamma Function, we have:

$$\begin{aligned}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right) &= \lim_{m\to\infty}\frac{\left(\sqrt{2\pi}\right)^n\left(\frac{mn}{e}\right)^{mn} m^{nz-n/2}m^{\sum_{k=0}^{n-1}k/n}}{(nz)\,(nz+1)\cdots(nz-1+mn)} \\ &= \lim_{m\to\infty}\frac{\left(\sqrt{2\pi}\right)^n\left(\frac{mn}{e}\right)^{mn} m^{nz-1/2}}{(nz)\,(nz+1)\cdots(nz-1+mn)} \\ &= \lim_{m\to\infty}\frac{\left(\sqrt{2\pi}\right)^n\left(\frac{m}{e}\right)^{m} m^{nz-1/2}n^{1/2-nz}}{(nz)\,(nz+1)\cdots(nz-1+m)} \\ &= \lim_{m\to\infty}\frac{\left(\sqrt{2\pi}\right)^{n-1} m!m^{nz-1}n^{1/2-nz}}{(nz)\,(nz+1)\cdots(nz-1+m)} \\ &= (2\pi)^{(n-1)/2}\, n^{1/2-nz}\,(nz-1)\,\Gamma\,(nz-1) \\ &= (2\pi)^{(n-1)/2}\, n^{1/2-nz}\Gamma\,(nz)\,.\end{aligned}$$

The proof is complete. □

2.2 Closed-Form Analytical Solution

Firstly, define $\tau := -\frac{1}{2}\sigma^{\alpha}\sec\frac{\alpha\pi}{2}(T-t)$. The Black-Scholes equation becomes the following:

$$\frac{\partial V}{\partial \tau} = (\gamma-1)\frac{\partial V}{\partial x} + \frac{\partial^{\alpha}V(x,\tau)}{\partial x^{\alpha}} - \gamma V$$

with boundary conditions $V(x,0) = \Pi(x)$ for Π defined previously and $\gamma = -\frac{2r}{\sigma^{\alpha}}\cos\frac{\alpha\pi}{2}$. γ can be reinterpreted as the relative interest rate with respect to volatility of fractional order. Taking the Fourier Transform on the above fractional partial differential equation yields the following:

$$\frac{\partial \hat{V}}{\partial \tau} = (\gamma-1)ik\frac{\partial \hat{V}}{\partial x} - |k|^{\alpha}\hat{V} - \gamma\hat{V}$$

with boundary conditions $\hat{V}(k,0) = \hat{\Pi}(k)$ where $\hat{\Pi}$ is the Fourier Transform of the payoff function Π. The above differential equation has the following solution:

$$\hat{V}(k,\tau) = e^{-\gamma\tau}\hat{\Pi}(k)\exp\left(-(1-\gamma)\tau ik - |k|^{\alpha}\tau\right).$$

Applying the convolution theorem for Fourier Transforms yields the following convolution,

$$V(x,\tau) = e^{-\gamma\tau}V(x,0) * \mathcal{F}^{-1}\{\exp\left(-(1-\gamma)\tau ik - |k|^{\alpha}\tau\right)\}$$

Define $P(x) := \mathcal{F}^{-1}\{e^{-|k|^{\alpha}\tau}\}$. Applying the shift theorem for Fourier transforms yields the following:

$$V(x,\tau) = e^{-\gamma\tau}V(x,0) * P(x-(1-\gamma)\tau)$$

By previous discussion, $e^{-|k|^{\alpha}}$ is the characteristic function of a centered and symmetric Levy distribution. Therefore Property 2.1 implies that the Fourier inversion of $e^{-|k|^{\alpha}\tau}$ is a multiple of the Levy stable density function $f_{\alpha,0}$. Applying Property 2.2 yields the following representation of P:

$$P(x) = \frac{1}{\tau^{1/\alpha}}f_{\alpha,0}\left(\frac{|x|}{\tau^{1/\alpha}}\right) = \frac{1}{\alpha\tau^{1/\alpha}}H^{1,1}_{2,2}\left[\frac{|x|}{\tau^{1/\alpha}}\left|\begin{matrix}(1-\frac{1}{\alpha},\frac{1}{\alpha}) & (\frac{1}{2},\frac{1}{2})\\ (0,1) & (\frac{1}{2},\frac{1}{2})\end{matrix}\right.\right].$$

Thus,

$$V(x,\tau) = \int_{-\infty}^{\infty} e^{-\gamma\tau}\Pi(k) * \frac{1}{\tau^{1/\alpha}}f_{\alpha,0}\left(\frac{|x-k-(1-\gamma)\tau|}{\tau^{1/\alpha}}\right)dk$$

Or equivalently,

$$V(x,\tau) = \int_{-\infty}^{\infty} e^{-\gamma\tau}\Pi(x-(1-\gamma)\tau-\tau^{1/\alpha}k)f_{\alpha,0}\left(|k|\right)dk.$$

Let $V_p(x,\tau)$ and $V_c(x,\tau)$ be the price for a European put option and a European call option respectively. For European put options, we have:

$$V_p(x,\tau) = Ke^{-\gamma\tau}\int_{d_1}^{\infty} f_{\alpha,0}\left(|k|\right)dk - e^{x}\int_{d_1}^{\infty}\exp\left(-\tau-\tau^{1/\alpha}k\right)f_{\alpha,0}\left(|k|\right)dk$$

where $d_1 = \dfrac{x-\log K-(1-\gamma)\tau}{\tau^{1/\alpha}}$. An analogous solution for European call options can be determined similarly. We will refer the above formula as the Black-Scholes Formula under the FMLS model. In the next section, we will discuss about the properties associated with this formula.

Recall that when $\alpha = 2$, the FMLS model degenerates to the traditional BS model. Furthermore, for boundary conditions $0 \leq t \leq T$ and $0 < S < \infty$ such that $C(S,t) \sim S$ as $S \to \infty$, and $C(S,T) = \max\{S-K, 0\}$ for strike price K, the solution of the Black-Scholes equation is provided below:

$$C(S,t) = SN(d_1) - Ke^{r(T-t)}N(d_2)$$

where,

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}s^2\right) ds$$

and,

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}} \\ d_2 &= \frac{\log(S/K) + (r - \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}} = d_1 - \beta\sqrt{T-t}. \end{aligned}$$

We will show that by setting $\alpha = 2$, our derived Black-Scholes formula degenerates into the standard Black-Scholes formula mentioned above.

Note that the Fox H-function is defined as a Mellin transform of a rational expression of the Gamma function. We will use this fact in the proof of the following theorem.

Theorem 2.2. *For European put options, the Black-Scholes formula under FMLS model degenerates to the Black-Scholes formula as $\alpha \to 2$. Equivalently,*

$$\lim_{\alpha \to 2} V_p(x,\tau) = Ke^{-\gamma\tau}N(-d_2) - e^x N(-d_1)$$

where $d_1 = \dfrac{x - \log K + (\gamma - 1)\tau}{\sqrt{2\tau}}$, $d_2 = d_1 - \sqrt{2\tau}$, *and*

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}s^2\right) ds.$$

Remark: The above variant is equivalent to the standard Black-Scholes formula. This is achieved by making the following substitutions: $r := \gamma$, $\beta := \sqrt{2}$, and $S := e^x$.

Proof. By Property 2.2, we have,

$$\begin{aligned} f_{2,0}(|k|) &= \lim_{\alpha \to 2} f_{\alpha,0}(|k|) \\ &= \lim_{\alpha \to 2} \frac{1}{\alpha} H^{1,1}_{2,2}\left[|m| \,\middle|\, \begin{matrix} (1-\frac{1}{\alpha},\frac{1}{\alpha}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{matrix} \right] \\ &= \frac{1}{2} H^{1,1}_{2,2}\left[|m| \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{2}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{matrix} \right] \\ &= \frac{1}{2} H^{1,0}_{1,1}\left[|m| \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{2}) \\ (0,1) \end{matrix} \right] \end{aligned}$$

where,

$$H^{1,0}_{1,1}(z) := H^{1,0}_{1,1}\left[z \,\middle|\, \begin{matrix} (a_1, A_1) \\ (b_1, B_1) \end{matrix} \right] = \frac{1}{2\pi i}\int_C \frac{\Gamma(b_1+B_1 s)}{\Gamma(a_1+A_1 s)} z^s ds.$$

The Fox H-function is defined as a Mellin transform of a rational expression of the Gamma function. By Definition 2.2, we see that $\mathcal{M}\{f_{2,0}(|k|)\} = \frac{1}{2}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}$.

Applying Lemma 2.1 for $n = 2$ yields:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = (2\pi)^{\frac{1}{2}} 2^{\frac{1}{2}-2z}\Gamma(2z)$$

or equivalently,

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Substituting $z = \frac{s}{2}$ yields,

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s).$$

Therefore,

$$\mathcal{M}\{f_{2,0}(|k|)\} = \frac{1}{2}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)} = \frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{4\sqrt{\pi}}.$$

Taking the inverse Mellin transform yields,

$$f_{2,0}(|k|) = \mathcal{M}^{-1}\left\{\frac{1}{2}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\right\} = \mathcal{M}^{-1}\left\{\frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{4\sqrt{\pi}}\right\} = \frac{e^{-k^2/4}}{2\sqrt{\pi}}.$$

This shows that $f_{2,0}(|k|)$ is identical to the Gaussian density function. Thus, by the Black-Scholes formula under FMLS model, with some algebraic manipulation, the Black-Scholes formula is obtained. Thus, the theorem is proved.

□

We can also determine the asymptotic behavior of our newly derived solution of the fractional partial differential equation. This will allow us to study the behavior of option pricing for extreme values. We will now propose the following theorem and provide proof.

Theorem 2.3. *For European put options, for $1 < \alpha \leq 2$, we have:*

$$\lim_{x\to-\infty} V_p(x,\tau) = Ke^{-\gamma\tau} \text{ and } \lim_{x\to\infty} V_p(x,\tau) = 0.$$

Proof. Note that as $x \to -\infty$, we have $d_1 \to -\infty$. For symmetric Levy density, we have $\int_{-\infty}^{\infty} f_{\alpha,0}(|k|)\,dk = 1$. Thus, we have

$$\begin{aligned}\lim_{x\to-\infty} V_p(x,\tau) &= Ke^{-\gamma\tau}\int_{-\infty}^{\infty} f_{\alpha,0}(|k|)\,dk - \lim_{x\to-\infty} e^x \int_{d_1}^{\infty} \exp\left(-\tau-\tau^{1/\alpha}k\right) f_{\alpha,0}(|k|)\,dk \\ &= Ke^{-\gamma\tau}.\end{aligned}$$

This proves the first statement of the theorem. To observe the behavior of the Black-Scholes equation under FMLS model for European puts as log price approaches infinity, we note that

$$\begin{aligned}\lim_{x\to\infty} V_p(x,\tau) &= Ke^{-\gamma\tau}\int_{\infty}^{\infty} f_{\alpha,0}(|k|)\,dk - \lim_{x\to\infty} e^x \int_{d_1}^{\infty} \exp\left(-\tau-\tau^{1/\alpha}k\right) f_{\alpha,0}(|k|)\,dk \\ &= -\lim_{x\to\infty}\frac{1}{e^{-x}}\int_{d_1}^{\infty} \exp\left(-\tau-\tau^{1/\alpha}k\right) f_{\alpha,0}(|k|)\,dk.\end{aligned}$$

Applying L'Hopital's Rule and using the fact that any density function approach zero as $x \to \infty$, we have,

$$\begin{aligned}\lim_{x\to\infty} V_p(x,\tau) &= K \lim_{x\to\infty} e^{-\tau\gamma}\tau^{-1/\alpha} f_{\alpha,0}\left(|d_1|\right) \\ &= K e^{-\tau\gamma}\tau^{-1/\alpha} \lim_{x\to\infty} f_{\alpha,0}\left(|d_1|\right) \\ &= 0\end{aligned}$$

since $d_1 \to \infty$ as $x \to \infty$. This completes the proof of the theorem. $\square$

The implications of Theorem 2.3 are expected. Under the FMLS model, a European put option is expected to hold the current value of the discounted strike price if asset price becomes really small. On the other hand, a European put option is expected to hold no value if the asset price increases to infinity.

In our derivation of the traditional Black-Scholes model, we specifically derived a partial differential equation that determines the behavior of European call options over maturity. Our derivation of the Black-Scholes equation applies to the behavior of European put options. It is our interest to determine if the relationship that lies between put option pricing and call option pricing satisfies the put-call parity. We will state and provide a proof of the following theorem.

Theorem 2.4. *For $1 < \alpha \leq 2$, under the same strike price K and maturity τ, the price of a European call option and the price of a European put option satisfies the* ***put-call parity****. Equivalently,*

$$V_c(x,\tau) - V_p(x,\tau) = e^x - Ke^{-\gamma\tau}.$$

Proof. Given that the price of both European call option and put option satisfies the Black-Scholes equation under FMLS model, for $V_{c-p}(x,\tau) := V_c(x,\tau) - V_p(x,\tau)$, we have,

$$\frac{\partial V_{c-p}}{\partial \tau} = (\gamma - 1)\frac{\partial V_{c-p}}{\partial x} + \frac{\partial^\alpha V_{c-p}(x,\tau)}{\partial x^\alpha} - \gamma V_{c-p} = 0$$

with boundary conditions $V_{c-p}(x,0) = e^x - K$ and $\gamma = -\frac{2r}{\sigma^\alpha}\cos\frac{\alpha\pi}{2}$. Using the same approach highlighted in our derivation of the Black-Scholes formula, we have

$$V_{c-p}(x,\tau) = -Ke^{-\gamma\tau}\int_{-\infty}^{\infty} f_{\alpha,0}\left(|k|\right) dk + e^x \int_{-\infty}^{\infty} \exp\left(-\tau - \tau^{1/\alpha}k\right) f_{\alpha,0}\left(|k|\right) dk.$$

In the proof of Theorem 2.3, we have established that $\int_{-\infty}^{\infty} f_{\alpha,0}\left(|k|\right) dk = 1$. Furthermore, upon realizing that the second integral is the Fourier transform of $f_{\alpha,0} = \mathcal{F}^{-1}\{e^{-|k|^\alpha}\}$, we have:

$$\begin{aligned} e^x \int_{-\infty}^{\infty} \exp\left(-\tau - \tau^{1/\alpha}k\right) f_{\alpha,0}\left(|k|\right) dk &= e^{x-\tau}\int_{-\infty}^{\infty} \exp\left(-i\left(-i\tau^{1/\alpha}\right)k\right) f_{\alpha,0}\left(|k|\right) dk \\ &= e^{x-\tau}\hat{f}_{\alpha,0}\left(-i\tau^{1/\alpha}\right) \\ &= e^{x-\tau+[i(-i\tau^{1/\alpha})]^\alpha} = e^x. \end{aligned}$$

Thus, we have,

$$V_{c-p}(x,\tau) = e^x - Ke^{-\gamma\tau}.$$

And the theorem is proven.

□

In this paper, we have proposed the Black-Scholes formula to determines the price of a European call option. To provide consistency of this paper, we will conclude this chapter with the generalized analytic form of the Black-Scholes formula for European call options by applying the previous theorem.

The Black-Scholes formula for European call options is provided as follows: for log price of a European call option x and time $\tau = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}(T-t)$ satisfying $0 \leq t \leq T$, the price of the European call option is analytically given as follows:

$$V_c(x,\tau) = e^x \int_{-\infty}^{d_1} \exp\left(-\tau - \tau^{1/\alpha}k\right) f_{\alpha,0}\left(|k|\right) dk - Ke^{-\gamma\tau}\int_{-\infty}^{d_1} f_{\alpha,0}\left(|k|\right) dk$$

where $d_1 = \frac{x - \log K - (1-\gamma)\tau}{\tau^{1/\alpha}}$ and $\gamma = -\frac{2r}{\sigma^\alpha}\cos\frac{\alpha\pi}{2}$ and

$$f_{\alpha,0}\left(|k|\right) = \frac{1}{\alpha} H_{2,2}^{1,1}\left[|k| \middle| \begin{array}{cc} (1-\frac{1}{\alpha},\frac{1}{\alpha}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{array} \right]$$

when $H_{2,2}^{1,1}$ is the Fox H-function defined in this section.

Remark: Note that by put-call parity, we have,

$$\int_{-\infty}^{\infty} \exp\left(-\tau - \tau^{1/\alpha} k\right) f_{\alpha,0}\left(|k|\right) dk = 1.$$

Chapter 3

Numerical Method

In this chapter, we will use the Finite Difference Method (FDM) to determine solutions for the Black-Scholes Equation under the FMLS model. We will also propose theorems that highlight the stability and the convergence of the method.

Let $V(x,t)$ be the price of a European vanilla option with respect to log price x and time t. Let r be the risk-free rate of interest, $\sigma > 0$ be the volatility, and $1 < \alpha \leq 2$ be the fractional parameter. Recall that the Black-Scholes Equation is given as follows:

$$\frac{\partial V(x,t)}{\partial t} + \left(r + \frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}\right)\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}\frac{\partial^\alpha V(x,t)}{\partial x^\alpha} = rV(x,t)$$

where,

$$\frac{\partial^\alpha V(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x} V(u,t)(x-u)^{-\alpha}du.$$

The boundary conditions are given as

$$V(x,T) = \Pi(x) := \begin{cases} \max\{e^x - K, 0\} & \text{for European call option} \\ \max\{K - e^x, 0\} & \text{for European put option} \end{cases}$$

where K is the strike price.

In Chapter 2, we have applied the following change of variables:

$$\tau := -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}(T-t).$$

This resulted in the following variant of the Black-Scholes Equation:

$$\frac{\partial V}{\partial \tau} = (\gamma - 1)\frac{\partial V}{\partial x} + \frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} - \gamma V$$

with boundary conditions $V(x,0) = \Pi(x)$ for Π defined above and $\gamma = -\frac{2r}{\sigma^\alpha}\cos\frac{\alpha\pi}{2}$. Throughout this chapter, we will solve this variant since this will simplify conditions for the finite difference method that will be specified in this chapter.

Note that the log price of the asset x satisfies $-\infty < x < \infty$. To apply the finite difference method, we truncate the interval and assume that for some constant $\varphi > 0$, the log price x satisfies $-\varphi < x < \varphi$. The constant φ will depend on the maximum price of the asset and can be readjusted appropriately.

3.1 Stability of Methods with Standard Grunwald Approximation

In this section, we develop a scheme using standard Grunwald weights to approximate the derivative of fractional order and establish that by using this approximation, both implicit and explicit Euler methods yield unstable results. [8]

Let $h = \frac{2\varphi}{m}$, $x_i = -\varphi + ih$ for $i = 0, \cdots, m$, let $\tau_n = n\Delta\tau$ where $\Delta\tau$ represents the incremental change of time in the interval $0 \le \tau \le -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}T$ for $n = 0, \cdots, m$, and $V_i^n = V(x_i, \tau_n)$.

The discrete Grunwald approximation for the fractional derivative is given by the formula:

$$\frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{m\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{t} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V(x - kh, \tau).$$

To simplify notation, let $g_k := \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$ be the normalized Grunwald weights. We analyze both forms of the Euler method to determine if the methods are stable.

Theorem 3.1. *The explicit Euler method of the Black-Scholes Eqauation under FMLS model with the discrete Grunwald approximation to the fractional derivative is unstable.*

Proof. Under the explicit Euler method, the Black-Scholes Equation reverts to the following: for $i = 1, \cdots, m-1$, and $n = 1, \cdots, m-1$,

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = -\gamma V_i^n + (\gamma - 1)\frac{V_i^n - V_{i-1}^n}{h} + \frac{1}{h^\alpha}\sum_{k=0}^{i} g_k V_{i-k}^n$$

Solving for V_i^{n+1} yields,

$$V_i^{n+1} = \mu V_i^n - \left(\frac{\gamma-1}{h} + \frac{\alpha}{h^\alpha}\right)\Delta\tau V_{i-1}^n + \frac{\Delta\tau}{h^\alpha}\sum_{k=2}^{i} g_k V_{i-k}^n - \gamma V_i^n \Delta\tau$$

where $\mu = 1 + \frac{\Delta\tau}{h}(\gamma - 1) + \frac{\Delta t}{h^\alpha}$. Assume that the following numerical value only have error, i.e. $\hat{V}_i^0 = V_i^0 + \epsilon_i^0$ for some $\epsilon_i^0 > 0$ dependent on i. This error produces a perturbation on the numerical value of $\hat{V}_i^1 = V_i^1 + \epsilon_i^1$. Then, we have

$$\hat{V}_i^1 = \mu\hat{V}_i^0 - \left(\frac{\gamma-1}{h} + \frac{\alpha}{h^\alpha}\right)\Delta\tau V_{i-1}^n + \frac{\Delta\tau}{h^\alpha}\sum_{k=2}^{i} g_k V_{i-k}^n - \gamma V_i^n \Delta\tau$$

$$= \mu(V_i^0 + \epsilon_i^0) = V_i^1 + \mu\epsilon_i^0$$

This implies that $\epsilon_i^1 = \mu\epsilon_i^0$. Recursively, we can deduce that $\epsilon_i^n = \mu^n\epsilon_i^0$. For the explicit Euler method to be stable, we require the error to be sufficiently small. Thus, μ must satisfy $|\mu| \le 1$. Note that,

$$1 < \left|1 + \frac{\Delta\tau}{h}(\gamma - 1) + \frac{\Delta\tau}{h^\alpha}\right| = |\mu|.$$

This implies that the explicit Euler method is unstable. The theorem is proven.

□

Theorem 3.2. *The implicit Euler method of the Black-Scholes Eqauation under FMLS model with the discrete Grunwald approximation to the fractional derivative is unstable.*

Proof. Under the implicit Euler method, the Black-Scholes Equation reverts to the following: for $i = 1, \cdots, m-1$ and $n = 1, \cdots, m-1$,

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = -\gamma V_i^{n+1} + (\gamma - 1)\frac{V_i^{n+1} - V_{i-1}^{n+1}}{h} + \frac{1}{h^\alpha}\sum_{k=0}^{i} g_k V_{i-k}^{n+1}$$

Solving for V_i^{n+1} yields,

$$V_i^{n+1} = \nu V_i^n - \nu \left(-\gamma V_i^{n+1} - \frac{\gamma - 1}{h} V_{i-1}^{n+1} + \frac{1}{h^\alpha} \sum_{k=0}^{i} g_k V_{i-k}^{n+1} \right) \Delta\tau$$

where $\nu = \left(1 - \frac{\Delta\tau}{h}(\gamma - 1) - \frac{\Delta t}{h^\alpha}\right)^{-1}$. Again, assume that the following numerical value only have error, i.e. $\hat{V}_i^0 = V_i^0 + \epsilon_i^0$ for some $\epsilon_i^0 > 0$ dependent on i. This error produces a perturbation on the numerical value of $\hat{V}_i^1 = V_i^1 + \epsilon_i^1$. Then, we have

$$\begin{aligned}\hat{V}_i^1 &= \nu \hat{V}_i^0 - \nu \left(-\gamma V_i^{n+1} - \frac{\gamma - 1}{h} V_{i-1}^{n+1} + \frac{1}{h^\alpha} \sum_{k=0}^{i} g_k V_{i-k}^{n+1} \right) \Delta\tau \\ &= \nu (V_i^0 + \epsilon_i^0) = V_i^1 + \nu \epsilon_i^0\end{aligned}$$

This implies that $\epsilon_i^1 = \nu \epsilon_i^0$. Recursively, we can deduce that $\epsilon_i^n = \nu^n \epsilon_i^0$. For the explicit Euler method to be stable, we require the error to be sufficiently small. Thus, ν must satisfy $|\nu| \leq 1$. Note that,

$$1 < \left| \left(1 - \frac{\Delta\tau}{h}(\gamma - 1) - \frac{\Delta t}{h^\alpha}\right)^{-1} \right| = |\nu|.$$

This implies that the implicit Euler method is unstable. The theorem is proven.

□

Therefore, both explicit and implicit Euler methods yield unstable results and the discrete Grunwald approximation cannot be used to solve the Black-Scholes equation under the FMLS model. In the next section, we will make a slight adjustment to the Grunwald approximation of the fractional derivative that will allow the Euler method to be consistent and unconditionally stable.

3.2 Stability of Methods with Shifted Grunwald Approximation

We will begin this section by defining the shifted Grunwald approximation of the fractional derivative. With the same conditions as before, we define the shifted Grunwald approximation as follows: for positive integer p,

$$A_hV(x,\tau) = \frac{1}{\Gamma(-\alpha)}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)}V(x-(k-p)h,\tau).$$

Furthermore, let $AV(x,\tau) = \dfrac{\partial^\alpha V(x,\tau)}{\partial x^\alpha}$ be the standard Grunwald approximation of the fractional derivative. We claim the following theorem:

Theorem 3.3. *As $h \to 0$, $A_hV(x,\tau) = AV(x,\tau) + O(h)$.*

Proof. Taking the Fourier transform of the definition of the shifted Grunwald approximation, we have,

$$\begin{aligned}
\hat{A_h V}(k,\tau) &= \frac{1}{h^\alpha}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{ik(m-p)h}\hat{V}(k,\tau)\\
&= \frac{1}{h^\alpha}e^{ikhp}(1-e^{ikh})^\alpha\hat{V}(k,\tau)\\
&= \frac{1}{h^\alpha}(-ikh)^\alpha\left(\frac{1-e^{ikh}}{-ikh}\right)^\alpha e^{ikhp}\hat{V}(k,\tau)\\
&= (-ik)^\alpha w(-ikh)\hat{V}(k,\tau)
\end{aligned}$$

with $w(z) = e^{zp}\left(\dfrac{1-e^{-z}}{z}\right)^\alpha = 1 - \left(p - \dfrac{\alpha}{2}\right)z + O(|z|^2)$. Since for all z, there exists $C > 0$ such that $|w(-iz) - 1| \leq C|z|$, we have,

$$\begin{aligned}
\hat{A_h V}(k,\tau) &= (-ik)^\alpha\hat{V}(k,\tau) + (-ik)^\alpha(w(-ikh)-1)\hat{V}(k,\tau)\\
&= \hat{AV}(k,\tau) + \hat{\phi}(h,k,\tau)
\end{aligned}$$

when $\hat{\phi}(h,k,\tau) = (-ik)^\alpha(w(-ikh)-1)\hat{V}(k,\tau)$. This implies that

$$|\hat{\phi}(h,k,\tau)| = |k|^\alpha C|hk||\hat{V}(k,\tau)|$$

Applying Theorem D.2 implies that as $h \to 0$, $A_hV(x,\tau) = AV(x,\tau) + O(h)$.

□

Theorem 3.3 discusses how as $h \to 0$, the shifted Grunwald approximation behaves similarly to the standard Grunwald approximation of the fractional derivative. This enables us to consider finite difference methods using this altered approximation. We will propose a theorem stating our observations.

Theorem 3.4. *Let $1 < \alpha \le 2$. Then the implicit Euler method of the Black-Scholes Equation under FMLS model with the shifted Grunwald approximation, given as follows,*

$$\frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{m\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{m} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V(x-(k-1)h,\tau)$$

for $h = \dfrac{2\varphi}{m}$, to the fractional derivative is consistent and unconditionally stable.

Proof. Note that there exists log price C such that $V(C,\tau) = 0$. Let $-\varphi < C$. The initial condition $V(-\varphi,\tau) = 0$ implies that for all $x \le -\varphi$, we have $V(x,\tau) = 0$. Theorem 3.3 implies that the truncation error of the shifted approximation is $O(h)$. This implies that the implicit method is consistent.

Using the above approximation, we have that for $i = 1, \cdots, m-1$ and $n = 1, \cdots, m-1$,

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = -\gamma V_i^{n+1} + (\gamma-1)\frac{V_i^{n+1} - V_{i-1}^{n+1}}{h} + \frac{1}{h^\alpha}\sum_{k=0}^{i} g_k V_{i-k+1}^{n+1}$$

After re-expressing the above equality, this creates a system of linear equations, which can be rewritten as the following matrix equation:

$$AV^{n+1} = V^n + \Delta t F^n$$

where $V^{n+1} = \left[V_0^{n+1}, V_1^{n+1}, V_2^{n+1}, ..., V_h^{n+1}\right]^T$,

$$V^n + \Delta\tau F^n = [0, (1-\gamma\Delta\tau)V_1^n, (1-\gamma\Delta\tau)V_2^n, ..., V(\varphi,\tau)]^T$$

and $A = [A_{i,j}]$ is the coefficient matrix, given as follows:

$$A_{i,j} = \begin{cases} 0 & j \ge i+2 \\ -g_0\frac{\Delta\tau}{h^\alpha} & j = i+1 \\ 1 + \frac{\Delta\tau}{h} - g_1\frac{\Delta\tau}{h^\alpha} & j = i \\ -\frac{\Delta\tau}{h} - g_2\frac{\Delta\tau}{h^\alpha} & j = i-1 \\ -g_{i-j+1}\frac{\Delta\tau}{h^\alpha} & j \le i-1 \end{cases}$$

Let λ be the eigenvalue of matrix A, so that for some nonzero vector X, we have $AX = \lambda X$. Choose index i so that $|x_i|$ is maximized where x_i represents the ith entry in X for $i = 0, ..., m$. Therefore, we have $\sum_{j=0}^{m} A_{i,j}x_j = \lambda x_i$, which implies,

$$\lambda = A_{i,i} + \sum_{j=0;j\neq i}^{m} A_{i,j}\frac{x_j}{x_i}.$$

Thus,

$$\lambda = \begin{cases} 1 + \frac{\Delta\tau}{h}\left(1 - \frac{x_{i-1}}{x_i}\right) - \frac{\Delta\tau}{h^\alpha}\left[g_1 + \sum_{j=0;j\neq 1}^{i+1} g_{i-j+1}\left|\frac{x_j}{x_i}\right|\right] & i \neq \{0, m\} \\ 1 & i = \{0, m\} \end{cases}$$

Since $\left|\frac{x_j}{x_i}\right| \leq 1$ and $g_j \geq 0$ for any $j = 0, 2, ...$, we have

$$\sum_{j=0,j\neq i}^{i+1} g_{i-j+1}\left|\frac{x_j}{x_i}\right| \leq \sum_{j=0,j\neq i}^{i+1} g_{i-j+1} \leq -g_1$$

and,

$$g_1 + \sum_{j=0;j\neq 1}^{i+1} g_{i-j+1}\left|\frac{x_j}{x_i}\right| \leq 0$$

This implies that all eigenvalues of A satisfy $|\lambda| \geq 1$. A is an invertible matrix so A^{-1} exists. The eigenvalues of A^{-1} satisfy $|\tilde{\lambda}| \leq 1$ and therefore, the spectral radius of A^{-1} satisfies $\rho(A^{-1}) \leq 1$. Note that the error vector ϵ^0 of V^0 results in the error vector ϵ^1 of V^1, related by $\epsilon^1 = A^{-1}\epsilon^0$. This implies that $||\epsilon^1|| \leq ||\epsilon^0||$. The method is unconditionally stable and the theorem is proven.

□

In the next section, we will apply the Finite Difference Method using the shifted Grunwald approximation of the fractional derivative. We will mimic the mathematical contents presented in Meerschaert and Tadjeran for European vanilla option pricing. [9]

Let $h = \dfrac{2\varphi}{m}$, $x_i = -\varphi + ih$ for $i = 0, \cdots, m$, let $\tau_n = n\Delta\tau$ where $\Delta\tau$ represents the incremental change of time in the interval $0 \le \tau \le -\dfrac{1}{2}\sigma^\alpha \sec\dfrac{\alpha\pi}{2}T$ for $n = 0, \cdots, m$, and $V_i^n = V(x_i, \tau_n)$.

The shifted Grunwald approximation for the fractional derivative is given by the formula:

$$\frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{m\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{m} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} V(x-(k-1)h, \tau).$$

Again, we will let $g_k := \dfrac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$ be the normalized Grunwald weights. Let $s(x,\tau) := (\gamma-1)\dfrac{\partial V}{\partial x} - \gamma V$ be the source/sink term of the Black-Scholes Equation. Under the interval $-\varphi < x < \varphi$, the finite difference method implies that

$$s_i^n := s(x_i, \tau_n) = (\gamma-1)\frac{V_{i+1}^n - V_i^n}{h} - \gamma V_i^n + O(h).$$

Therefore, by using the shifted Grunwald approximation, we see that the Black-Scholes Equation under the finite difference method is given as:

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \frac{1}{h^\alpha} \sum_{k=0}^{i+1} g_k V_{i-k+1}^n + s_i^n.$$

Solving for V_i^{n+1} yields that for $-\varphi < x < \varphi$, we have

$$V_i^{n+1} = \frac{\Delta\tau}{h^\alpha} g_0 V_{i+1}^n + \left(1 + \frac{\Delta\tau}{h^\alpha} g_1\right) V_i^n + \frac{\Delta\tau}{h^\alpha} \sum_{k=2}^{i+1} g_k V_{i-k+1}^n + s_i^n \Delta\tau$$

with $h = \dfrac{2\varphi}{m}$ and s_i^n defined above. For $i = 1, \cdots, m-1$ and for each $n = 1, \cdots, m-1$, the $m-1$ equalities degenerate into a matrix equation given below:

$$\begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ V_3^{n+1} \\ \vdots \\ V_{m-1}^{n+1} \end{bmatrix} = \left(I + \frac{\Delta\tau}{h^\alpha} \begin{bmatrix} g_1 & g_0 & 0 & \cdots & 0 \\ g_2 & g_1 & g_0 & \cdots & 0 \\ g_3 & g_2 & g_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{m-1} & g_{m-2} & g_{m-3} & \cdots & g_1 \end{bmatrix} \right) \begin{bmatrix} V_1^n \\ V_2^n \\ V_3^n \\ \vdots \\ V_{m-1}^n \end{bmatrix} + \frac{\Delta\tau}{h^\alpha} V_0^n \begin{bmatrix} g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_m \end{bmatrix} + \Delta\tau \vec{s_n}$$

with

$$\vec{s_n} = \frac{1}{h}\begin{bmatrix} 1-\gamma-h\gamma & \gamma-1 & 0 & \cdots & 0 \\ 0 & 1-\gamma-h\gamma & \gamma-1 & \cdots & 0 \\ 0 & 0 & 1-\gamma-h\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\gamma-h\gamma \end{bmatrix}\begin{bmatrix} V_1^n \\ V_2^n \\ V_3^n \\ \vdots \\ V_{m-1}^n \end{bmatrix} + \frac{\gamma-1}{h}V_m^n\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Solving the above matrix equation using Gaussian Elimination requires a computational cost of $O(m^3)$ operations and a storage cost of $O(m^2)$ per time step. Note that for all $n = 0, \cdots, m$,

$$V_0^n = \begin{cases} 0 & \text{for European call option} \\ Ke^{-\gamma\tau} & \text{for European put option} \end{cases}$$

and,

$$V_m^n = \begin{cases} e^{\varphi} & \text{for European call option} \\ 0 & \text{for European put option} \end{cases}$$

Also, note that for all $i = 0, \cdots, m$,

$$V_i^0 = \begin{cases} \max\{e^{x_i} - K, 0\} & \text{for European call option} \\ \max\{K - e^{x_i}, 0\} & \text{for European put option} \end{cases}$$

In the next chapter, we will present an efficient finite difference method to solve the Black-Scholes equation under FMLS model. We will attempt to reduce the storage and computational cost of the method. In doing so, we must add a necessary restriction to the initial conditions inconveniently not satisfied by the problem as stated.

Chapter 4

Fast Numerical Method

Under the standard finite difference method, recall that the price of a European vanilla option $V(x,\tau)$ satisfies the following recursion: for $-\varphi < x < \varphi$, we have

$$V_i^{n+1} = \frac{\Delta\tau}{h^\alpha} g_0 V_{i+1}^n + \left(1 + \frac{\Delta\tau}{h^\alpha} g_1\right) V_i^n + \frac{\Delta\tau}{h^\alpha} \sum_{k=2}^{i+1} g_k V_{i-k+1}^n + s_i^n \Delta\tau$$

with $h = \dfrac{2\varphi}{m}$ and,

$$s_i^n := s(x_i, \tau_n) = (\gamma - 1)\frac{V_{i+1}^n - V_i^n}{h} - \gamma V_i^n + O(h).$$

The above recursion creates a system of equations which can be re-expressed as a matrix equation. Under the standard finite difference method, the matrix equation requires a storage of $O(m^2)$ and a computational cost of $O(m^3)$. In this section, we propose a more efficient method by reducing the computational cost in the inversion of the coefficient matrix. The result of this inversion method reduces the storage cost from $O(m^2)$ to $O(m)$ and reduces the computational cost to $O(m \log m)$ at each iteration.

In this chapter, we will discuss about the limitations of the direct $O(m \log^2 m)$ finite difference method and propose a fast locally conservative finite volume method that similarly reduces the storage cost and computational cost of the standard finite difference method. In using this method, we require to re-express our equation in a conservative way.

Let $V(x,t)$ be the price of the European vanilla option with respect to log price x and time t. Let r be the risk-free rate of interest, $\sigma > 0$ be the volatility, and $1 < \alpha \leq 2$ be the fractional parameter. Recall that after a change of variables, for $-\infty < x < \infty$, the FMLS Fractional Black-Scholes Equation is given as follows:

$$\frac{\partial V}{\partial \tau} = (\gamma - 1)\frac{\partial V}{\partial x} + \frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} - \gamma V$$

with boundary conditions $V(x,0) = \Pi(x)$ for $\Pi(x)$ representing the payoff function for European vanilla options defined previously and $\gamma = -\frac{2r}{\sigma^\alpha}\cos\frac{\alpha\pi}{2}$.

For a direct $O(m \log^2 m)$ finite difference method approach, we require the extreme values of the log price to take equal values. [13] Note that for maturity τ and log price x, the price of a European call option $V(x,\tau)$ under the FMLS model is monotonic. Specifically, recall the analytic solution for European call options under the FMLS model: for log price of a European call option x and time $\tau = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}(T-t)$ satisfying $0 \leq t \leq T$, the price of the European call option is analytically given as follows:

$$V_c(x,\tau) = e^x \int_{-\infty}^{d_1} \exp\left(-\tau - \tau^{1/\alpha}k\right) f_{\alpha,0}\left(|k|\right) dk - Ke^{-\gamma\tau}\int_{-\infty}^{d_1} f_{\alpha,0}\left(|k|\right) dk$$

where $d_1 = \dfrac{x - \log K - (1-\gamma)\tau}{\tau^{1/\alpha}}$ and $\gamma = -\dfrac{2r}{\sigma^\alpha}\cos\dfrac{\alpha\pi}{2}$ and

$$f_{\alpha,0}\left(|k|\right) = \frac{1}{\alpha}H_{2,2}^{1,1}\left[|k| \left| \begin{matrix} (1-\frac{1}{\alpha},\frac{1}{\alpha}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{matrix} \right.\right]$$

when $H_{2,2}^{1,1}$ is the Fox H-function.

Taking the derivative with respect to x yields:

$$\begin{aligned}\frac{\partial V_c}{\partial x} &= e^x \exp\left(-\tau - \tau^{1/\alpha} d_1\right) f_{\alpha,0}(|d_1|)\frac{1}{\tau^{1/\alpha}} + e^x \int_{-\infty}^{d_1} \exp\left(-\tau - \tau^{1/\alpha} k\right) f_{\alpha,0}\left(|k|\right) dk \\ &\quad - Ke^{-\gamma\tau} f_{\alpha,0}(|d_1|)\frac{1}{\tau^{1/\alpha}} \\ &= e^x \int_{-\infty}^{d_1} \exp\left(-\tau - \tau^{1/\alpha} k\right) f_{\alpha,0}\left(|k|\right) dk > 0.\end{aligned}$$

By Theorem 2.4, we have,

$$\frac{\partial V_c}{\partial x} - \frac{\partial V_p}{\partial x} = e^x$$

Or,

$$\frac{\partial V_p}{\partial x} = -e^x \int_{d_1}^{\infty} \exp\left(-\tau - \tau^{1/\alpha} k\right) f_{\alpha,0}\left(|k|\right) dk < 0.$$

Therefore, we have shown that if V represents the price of a European call option, $V(x, \tau)$ is monotonically increasing with respect to log price x. Similarly, if V represents the price of a European put option, $V(x, \tau)$ is monotonically decreasing with respect to log price x.

However, it is not possible to truncate the tails so that there exist two endpoints with equal values. If there exists L and R so that $V(L, \tau) = V(R, \tau)$, by Rolle's Theorem, there exists a relative minimum or maximum in the interval $L < x < R$. This is a contradiction since the European option value function is a monotonic function.

A naive approach to circumvent this issue is to study the Taylor series representation of $V(x, \tau)$. This will require the Taylor series representation for Levy stable density functions. In general, the Levy stable distribution can be expressed as the real part of a simpler integral given below:

$$f(x; \alpha, \beta, c, \mu) = \frac{1}{\pi}\text{Re}\left[\int_0^{\infty} \exp\left(it(x-\mu) - (ct)^{\alpha}\left(1 - i\beta\tan\frac{\alpha\pi}{2}\right)\right)\right]$$

By expressing $\exp\left(-(ct)^{\alpha}\left(1 - i\beta\tan\frac{\alpha\pi}{2}\right)\right)$ as a Taylor series and reversing the order of integration and summation yields,

$$f(x;\alpha,\beta,c,\mu) = \frac{1}{\pi}\mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(-q)^n}{n!}\left(\frac{i}{x-\mu}\right)^{\alpha n+1}\Gamma(\alpha n+1)\right]$$

which is valid for all $x \neq \mu$ where $q := c^{\alpha}(1 - i\beta\tan\frac{\alpha\pi}{2})$. Thus,

$$f_{\alpha,0}(x) = f(x;\alpha,0,1,0) = \frac{1}{\pi}\mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{i}{x}\right)^{\alpha n+1}\Gamma(\alpha n+1)\right]$$

Or,

$$f_{\alpha,0}(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(\alpha n+1)}{n!}x^{-\alpha n-1}\mathrm{Re}\left[i^{\alpha n+1}\right].$$

Note that,

$$\mathrm{Re}\left[i^{\alpha n+1}\right] = \mathrm{Re}\left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha n+1}\right] = \cos\left(\frac{\pi\alpha n}{2} + \frac{\pi}{2}\right) = -\sin\frac{\pi\alpha n}{2}.$$

Thus,

$$f_{\alpha,0}(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\Gamma(\alpha n+1)}{n!}\sin\left(\frac{\pi\alpha n}{2}\right)x^{-\alpha n-1}.$$

For general Levy stable density function, we have the following asymptotic:

$$f(x;\alpha,\beta,c,\mu) \sim \frac{1}{\pi}c^{\alpha}(1+\mathrm{sgn}(x)\beta)\sin\frac{\pi\alpha}{2}\Gamma(\alpha+1)|x|^{-1-\alpha}.$$

Therefore,

$$f_{\alpha,0}(x) \sim \frac{1}{\pi}\sin\frac{\pi\alpha}{2}\Gamma(\alpha+1)|x|^{-1-\alpha}.$$

However, in this naive approach, there are many limitations in the direct $O(m\log^2 m)$ method. In performing numerical experiments using the Taylor series representation of the closed-form solution $V_p(x,\tau)$, we cannot express the integrals in terms of standard built-in functions. Due to the series conditional convergence, the question of the accuracy of the numerical approximation also arises.

The low convergence rate at $x \to \infty$ of the Levy density function $f_{\alpha,0}(x)$ has the added difficulty of computing semi-infinite integrals in our closed form using any numerical methods. Although the generalized Laguerre-Gauss quadrature efficiently calculates integrals of this form, a naive truncation proposed in the Taylor series expansion, specifically the infinite series of the Grunwald-Letnikov fractional derivative, yields a finite difference method that is unconditionally unstable. The infinite series also does not accurately approximate put values for all x satisfying $x < Ke^{(1-\gamma)\tau}$ since the assumption of $d_1 > 0$ is necessary to define the Taylor series expansion of the Levy density function $f_{\alpha,0}(x)$ for all $1 < \alpha < 2$.

To avoid these issues, we will alternatively use fast finite volume methods which similarly use banded coefficient matrices to reduce the computational cost from $O(m^3)$ to $O(m \log m)$ and storage cost from $O(m^2)$ to $O(m)$. For the application of this method, we relieve the restrictive boundary conditions of $u(L,\tau) = u(R,\tau) = 0$ and re-express the Black-Scholes equation in a conservative way. This calls for a crucial definition and some additional discussion. More information can be found in Cheng, Wang, and Wang. [6]

4.2 Preliminaries for Fast Finite Volume Methods

Let $\alpha = 2 - \beta$ for $0 < \beta < 1$. The left and right **Caputo fractional derivative of order** $1 - \beta$ are respectively defined as follows:

$$ {}_aD_x^{-\beta} Dg(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-s)^{\beta-1} g'(s)ds $$

$$ {}_xD_b^{-\beta} Dg(x) = \frac{1}{\Gamma(\beta)} \int_x^b (s-x)^{\beta-1} g'(s)ds $$

Recall that after a change of variables, for $-\infty < x < \infty$, the FMLS Fractional Black-Scholes Equation is given as follows:

$$ \frac{\partial V}{\partial \tau} = (\gamma - 1)\frac{\partial V}{\partial x} + \frac{\partial^\alpha V(x,\tau)}{\partial x^\alpha} - \gamma V $$

with boundary conditions $V(x,0) = \Pi(x)$ for $\Pi(x)$ representing the payoff function for European vanilla options defined previously and $\gamma = -\frac{2r}{\sigma^\alpha}\cos\frac{\alpha\pi}{2}$.

Let $V(x,\tau) = \exp(-\gamma\tau)u(x,\tau)$. Then, we have the following:

$$\frac{\partial V}{\partial \tau} = \exp(-\gamma\tau)\frac{\partial u}{\partial \tau} - \gamma u \exp(-\gamma\tau)$$
$$\frac{\partial V}{\partial x} = \exp(-\gamma\tau)\frac{\partial u}{\partial x}$$

Furthermore, we have,

$$\frac{\partial^\alpha V}{\partial x^\alpha} := D_x^\alpha V(x,\tau) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1) D_x^{\alpha-n} u(x,\tau) D_x^n \exp(-\gamma\tau)}{\Gamma(\alpha-n+1)n!}$$
$$= D_x^\alpha u(x,\tau)\exp(-\gamma\tau).$$

Details of this proof can be referred to Osler. [10] Our partial differential equation becomes the following:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^\alpha u}{\partial x^\alpha} + (\gamma-1)\frac{\partial u}{\partial x}$$

Or equivalently,

$$\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x}\left({}_xD_\varphi^{-\beta} Du(x,\tau)\right) = (\gamma-1)\frac{\partial u}{\partial x}$$

The above variant of the Black-Scholes equation under the FMLS model is re-expressed in a form renowned as **divergence form**. This is the required form necessary to successfully apply the fast finite volume method. Unlike the direct $O(m\log^2 m)$ finite difference method, we do not require $u(-\varphi,\tau) = u(\varphi,\tau) = 0$.

Continuing our construction for this method, let us define a temporal partition on the interval $0 \leq \tau \leq -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}T$ by $\tau_n = n\Delta\tau$ for $n = 0, 1, ..., m$ with $\Delta\tau = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}\frac{T}{m}$. We will use the finite difference approximation of the time derivative to re-express the partial differential equation at time τ_n as follows:

$$\frac{u(x,\tau_n) - u(x,\tau_{n-1})}{\Delta\tau} - \frac{\partial}{\partial x}\left({}_xD_\varphi^{-\beta} u'(x,\tau_n)\right) = f(x,\tau_n)$$

where $f(x,\tau_n) := (\gamma-1)\frac{\partial u}{\partial x}(x,\tau_n)$ represents the source/sink term. Equivalently,

$$u(x,\tau_n) - \Delta\tau \frac{\partial}{\partial x}\left({}_xD_{\varphi}^{-\beta} u'(x,\tau_n)\right) = \Delta\tau f(x,\tau_n) + u(x,\tau_{n-1}).$$

Next, define a uniform spatial partition on the truncated interval of the log price $[-\varphi,\varphi]$ by letting $x_i = ih$ for $i = 0, 1, ..., m$ with $h = \frac{2\varphi}{m}$. Furthermore, define $x_{i-\frac{1}{2}} = (x_{i-1} + x_i)/2$ for $i = 1, ..., m$ to be the midpoint of the interval $[x_{i-1}, x_i]$. Then, by integrating the governing equation on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ for $i = 1, ..., m$ yields the following:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,\tau_n)dx - \Delta\tau\ {}_xD_{\varphi}^{-\beta} u'(x,\tau_n)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \Delta\tau \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,\tau_n)dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,\tau_{n-1})dx.$$

Let $S_h(-\varphi,\varphi)$ denote the space of continuous and piecewise-linear functions with respect to the spatial partition that vanishes at $x = -\varphi$ and $x = \varphi$. Define the nodal basis function $\phi_k(x)$ for $k = 1, ..., m-1$ as follows:

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & x \in [x_{k-1}, x_k] \\ \dfrac{x_{k+1} - x}{h} & x \in [x_k, x_{k+1}] \\ 0 & \text{elsewhere} \end{cases}$$

Let $u_k^n := u(x_k, \tau_n)$. Then, $u_h(x,\tau_n) \in S_h(-\varphi,\varphi)$ can be represented as:

$$u_h(x,\tau_n) = \sum_{k=1}^{m-1} u_k^n \phi_k(x).$$

Thus, for all $i = 1, ..., m-1$, the finite volume scheme is formulated as:

$$\sum_{j=1}^{m-1} u_j^n \left[\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \phi_j(x)dx - \Delta\tau\ {}_xD_{\varphi}^{-\beta} \phi_j'(x)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\right]$$
$$= \Delta\tau \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,\tau_n)dx + \sum_{j=1}^{m-1} u_j^{n-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \phi_j(x)dx.$$

Let $f_k^n := f(x_k, \tau_n)$ and define the vectors $u^n = [u_1^n, u_2^n, ..., u_{m-1}^n]^T$ and $f^n = [f_1^n, f_2^n, ..., f_{m-1}^n]^T$ be the vectors corresponding to the numerical approximation and the source/sink terms at time step τ_n. Let $M = [M_{i,j}]_{i,j=1}^{m-1}$ be the mass matrix and

$B^n = [B^n_{i,j}]_{i,j=1}^{m-1}$ be the stiffness matrix at time step τ_n. The entries of the matrices M and B^n and vector f^n are given as follows; for $1 \le i, j \le m-1$:

$$M_{i,j} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \phi_j(x)dx$$

$$B^n_{i,j} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} D_x D_\varphi^{-\beta} D\phi_j(x)dx$$

$$f^n_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,\tau_n)dx$$

We will shortly determine the entries for M and B^n. The entries of vector f^n are simplified as follows:

$$f^n_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,\tau_n)dx = (\gamma - 1)\, u(x,\tau_n)|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = (\gamma-1)\left(u^n_{i+\frac{1}{2}} - u^n_{i-\frac{1}{2}}\right).$$

The finite volume scheme can be re-expressed in the following matrix equation:

$$(M - \Delta\tau B^n)u^n = Mu^{n-1} + \Delta\tau f^n$$

The mass matrix M takes the same form as its counterpart for the classical diffusion equation. It is given as follows:

$$M = \frac{1}{8}h \begin{bmatrix} 6 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 6 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 6 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}$$

The difference between the finite volume scheme and the classical diffusion equation lies in the properties of stiffness matrix B^n. By observing the support of the differential operators, we can conclude that the stiffness matrix is a full matrix which requires a storage cost of $O(m^2)$ and a computational cost of $O(m^3)$. Many methods, including the finite volume method being discussed in this chapter, have been proposed to effectively reduce these costs.

Under algebraic manipulations, one can deduce that the entries of the stiffness matrix can be rewritten as follows:

$$B^n_{i,j} = \frac{1}{\Gamma(\beta+1)h^{1-\beta}} g^{(\beta)}_{i-j}$$

where $g^{(\beta)}_j$ is defined as follows:

$$g^{(\beta)}_j = \begin{cases} 3\left(-j-\frac{1}{2}\right)^\beta - 3\left(-j+\frac{1}{2}\right)^\beta & 2-m \le j \le -2 \\ 3\left(\frac{1}{2}\right)^\beta + \left(\frac{5}{2}\right)^\beta - 3\left(\frac{3}{2}\right)^\beta & j=-1 \\ \left(\frac{3}{2}\right)^\beta - 3\left(\frac{1}{2}\right)^\beta & j=0 \\ \left(\frac{1}{2}\right)^\beta & j=1 \\ 0 & 2 \le j \le m-2 \end{cases}$$

Alternatively, we can decompose the stiffness matrix as follows:

$$B^n = \frac{1}{\Gamma(\beta+1)h^{1-\beta}} G$$

where,

$$G = \begin{bmatrix} g^{(\beta)}_0 & g^{(\beta)}_1 & \cdots & g^{(\beta)}_{m-3} & g^{(\beta)}_{m-2} \\ g^{(\beta)}_{-1} & g^{(\beta)}_0 & \cdots & g^{(\beta)}_{m-4} & g^{(\beta)}_{m-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{(\beta)}_{3-m} & g^{(\beta)}_{4-m} & \cdots & g^{(\beta)}_0 & g^{(\beta)}_1 \\ g^{(\beta)}_{2-m} & g^{(\beta)}_{3-m} & \cdots & g^{(\beta)}_{-1} & g^{(\beta)}_0 \end{bmatrix}$$

Note that B^n does not depend on n. We will rename $B := B^n$ to emphasize this relationship between the stiffness matrix and the time step n. Our matrix equation now becomes:

$$\left(\frac{1}{\Delta\tau} M - B\right) u^n = \frac{1}{\Delta\tau} M u^{n-1} + f^n$$

Note that we can conveniently compute $\frac{1}{\Delta\tau}M$ as follows:

$$\frac{1}{\Delta\tau}M = \frac{1}{2}\frac{\varphi}{\sigma^{2-\beta}T}\cos\left(\frac{\pi\beta}{2}\right)\begin{bmatrix} 6 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 6 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & 6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 6 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}$$

Contrary to the standard requirement of $O(m^2)$ storage cost for any full matrix, the matrix B requires $O(m)$ storage.

4.3 Fast $O(m \log m)$ Algorithm for the Evaluation of Bu

In this section, we discuss a more efficient method of evaluating the matrix-vector multiplication of the stiffness matrix B and some vector u. Firstly, define the following matrix $\tilde{G}$ as follows:

$$\tilde{G} = \begin{bmatrix} 0 & g_{2-m}^{(\beta)} & \cdots & g_{-2}^{(\beta)} & g_{-1}^{(\beta)} \\ g_{m-2}^{(\beta)} & 0 & \cdots & g_{-3}^{(\beta)} & g_{-2}^{(\beta)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2}^{(\beta)} & g_{3}^{(\beta)} & \cdots & 0 & g_{2-m}^{(\beta)} \\ g_{1}^{(\beta)} & g_{2}^{(\beta)} & \cdots & g_{m-2}^{(\beta)} & 0 \end{bmatrix}$$

and define the following $(2m-2) \times (2m-2)$ circulant matrix as follows:

$$C_{2m-2} = \begin{bmatrix} G & \tilde{G} \\ \tilde{G} & G \end{bmatrix}.$$

The circulant matrix C_{2m-2} can be decomposed as follows

$$C_{2m-2} = F_{2m-2}^{-1}\text{diag}(F_{2m-2}c)F_{2m-2}$$

where c represents the first column vector of C_{2m-2} and F_{2m-2} represents the $(2m-2) \times (2m-2)$ discrete Fourier transform matrix with entries given by

$$F_{2m-2}(j,l) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\pi i j l}{m-1}\right) \qquad 0 \le j, l \le 2m-3$$

Given this, we describe the procedures necessary to evaluate this matrix-vector multiplication efficiently.

1. Rewrite u as $u_{2m-2} = [u^T, 0]^T$ so that,

$$C_{2m-2} v_{2m-2} = \begin{bmatrix} Gu \\ \tilde{G}u \end{bmatrix}$$

2. Similarly, evaluate the matrix-vector product $w_{2m-2} = F_{2m-2} u_{2m-2}$ in $O(m \log m)$ operations by using fast Fourier transform. w_{2m-2} is the discrete Fourier transform of u_{2m-2}.

3. Now, we evaluate the matrix-vector product $v_{2m-2} = F_{2m-2} c_{2m-2}$ in $O(m \log m)$ operations by using fast Fourier transform.

4. Next, evaluate the Hadamard product $z_{2m-2} = w_{2m-2} \cdot v_{2m-2} = [w_1 v_1, \cdots, w_{2n-2} v_{2n-2}]^T$ in $O(N)$ operations.

5. Lastly, evaluate $y_{2m-2} = F^{-1}_{2m-2} z_{2m-2}$ in $O(m \log m)$ operations using inverse fast Fourier transform. This yields:

$$y_{2m-2} = C_{2m-2} v_{2m-2} = \begin{bmatrix} Gu \\ \tilde{G}u \end{bmatrix}$$

6. Multiplying by the scalar $\dfrac{1}{\Gamma(\beta+1) h^{1-\beta}}$ to the vector y_{2m-2} in $O(m)$ operations yields Bu.

By the above procedures, it can be deduced that the computational cost of evaluating Bu for stiffness matrix B and some vector u is $O(m \log m)$. This is a significant improvement compared to the standard finite difference method's computational cost of $O(m^3)$ operations.

4.4 Fast Conjugate Gradient Square Method with $O(m)$ Storage

Since the stiffness matrix B is full, the direct method of computing the matrix-vector multiplication requires $O(m^3)$ per time step. Given that we have deduced an algorithm to efficiently calculate Bu, it motivates us to consider the conjugate gradient square method (CGS), a generalized method of the standard conjugate gradient method. This will allow us to accelerate the performance of the finite volume scheme.

We avoid using the standard conjugate gradient method as this method does not apply for nonsymmetric systems The residual vectors cannot be made orthogonal with short recurrences. The pseudocode for CGS on the finite volume scheme discussed throughout this chapter is provided below:

Algorithm 1 Conjugate Gradient Squared Method

procedure At each time step τ_n, choose $u^{(0)} = u^{n-1}$ and compute $r^{(0)} = f^n - A^n u^{(0)}$. Choose $\tilde{r}$ (for example, $\tilde{r} = r^{(0)}$)
  for $i = 1, 2, \cdots$ **do**
    $\rho_{i-1} = \tilde{r}^T r^{(i-1)}$
    if $\rho_{i-1} = 0$, the method fails
    if $i = 1$ **then**
      $w^{(1)} = r^{(0)}$
      $p^{(1)} = w^{(1)}$
    else
      $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
      $w^{(i)} = r^{(i-1)} + \beta_{i-1} q^{(i-1)}$
      $p^{(i)} = w^{(i)} + \beta_{i-1}(q^{(i-1)} + \beta_{i-1} p^{(i-1)})$
    $\hat{v} = A^n p^{(i)}$
    $\alpha_i = \rho_{i-1}/\tilde{r}^T \hat{v}$
    $q^{(i)} = w^{(i)} - \alpha_i \hat{q}$
    $u^{(i)} = u^{(i-1)} + \alpha_i (w^{(i)} + q^{(i)})$
    $\hat{q} = A^n (w^{(i)} + q^{(i)})$
    $r^{(i)} = r^{(i-1)} - \alpha_i \hat{q}$
    $\delta = ||f^n - A^n u^{(i)}||$
    check convergence; continue if necessary
  $u^n = u^{(i)}$

The above pseudocode requires $O(m^2)$ operations to compute the matrix-vector multiplication. By the previous section, it has been shown that the computational cost can be reduced to $O(m \log m)$ operations. Applying the fast algorithm discussed in Section 4.3 onto FCGS method yields a fast finite volume method that efficiently computes the numerical values of the price of European vanilla options.

Chapter 5

Numerical Experiments and Observations

In this chapter, we will carry out numerical experiments to investigate the performance of Euler's backward method for the finite volume scheme discussed in Chapter 4. We will also implement the Fast Conjugate Gradient Squared Method to accelerate the performance of the finite volume scheme and to observe its reduction of the CPU time. Lastly, we will simulate the Black-Scholes model under the FMLS process for European puts with different choices of α to emphasize the effects of the return distribution when compared to the underlying price.

5.1 Simulation of S&P 500 Options Market

Firstly, consider the one year time interval between September 15, 2014 to September 15, 2015. Using recently observed data from the S&P 500 options market in this time interval, we note that Chicago Board Options Exchange Volatility Index (VIX), a standard measure of implied volatility, consistently fluctuates between $\sigma = 0.1088$ and $\sigma = 0.4074$. For the purposes of our numerical experiments, we will let $\sigma = 0.1601822$ represent the average daily VIX observed in this time frame. The data can be observed and retrieved from YAHOO! Finance.

Since the 2007 financial crisis, the U.S. Federal Reserve have kept interest rates between 0% and 0.25% in attempts to stabilize the U.S. economy and the financial system. This consequentially creates unnecessary inflation on many European vanilla options. Recent trends have suggest the U.S. Federal Reserve to consider raising interest rates to reduce inflation. For the sake of our numerical experiments, we will

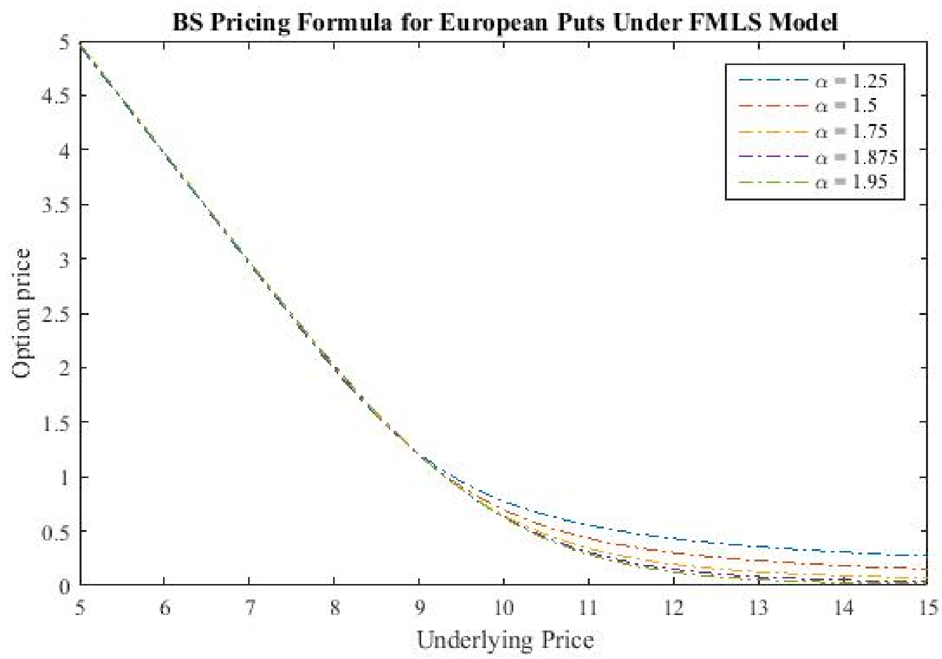

Figure 5.1: European puts at different α values. Model parameters are $K = \$10$, $r = 0.025$, and $\tau = 1$.

let $r = 0.0025$.

Lastly, we will assume that the maximum asset price is \$150. This suggests that we will truncate our log price interval to the following interval for both finite difference scheme and the finite volume scheme: $-5 < x < 5$. For our numerical experiments, we let $\varphi = 5$.

Under these realistic conditions, we are able to simulate the Black-Scholes equation under the FMLS model for varying parameters α. Provided above depicts the overall effect of European puts under different choices of α assuming that strike price $K = \$10$. In our simulation, we choose $\alpha = \{1.25, 1.5, 1.75, 1.875, 1.95\}$. As expected, as $\alpha \to 2$, the tail behavior of the return distribution becomes flat and the effect of leptokurtosis diminishes. Under put-call parity, we should also expect a similar behavior for prices of European call options.

We will also discuss the CPU time usage for the methods presented in this thesis. Under the finite volume scheme (FV), we consider a spatial and temporal partition defined by the midpoints of consecutive nodes and solve the Black-Scholes equation under FMLS under these points using Gaussian Elimination. Furthermore, we will implement a Fast Conjugate Gradient Squared acceleration (FCGS) onto the finite volume scheme to reduce the CPU time. The results are compiled in the following table.

Table 5.1: CPU Time Usage of Numerical Methods.

m	FV with GE	FV with FCGS
2^9	0.094 s	0.016 s
2^{10}	0.4414 s	0.153 s
2^{11}	1.706 s	0.478 s
2^{12}	7.233 s	1.235 s

As noted in Table 5.1, it is expected that the finite volume scheme with Gaussian elimination requires more CPU time to successfully determine the prices of European vanilla options under any choice of α. Using various choices of m, we can observe that the times increase significantly under this method. Recall that we require a computational cost of $O(m^3)$ and a storage cost of $O(m^2)$ operations. This suggests that the CPU time increases exponentially as we scale m as observed in the table above.

When applying the Fast Conjugate Gradient Squared method, we successfully reduced the computational cost to respectively $O(m \log m)$ operations per iterate. Thus, we should expect that the CPU time reduces for all choices of m. Again, this can be observed by the results compiled in the table above. Under this acceleration, we reduce the storage cost from $O(m^2)$ to $O(m)$. We should expect a reduction of CPU time since the finite volume scheme under this method does not require much storage. This again can be observed in the data presented in Table 5.1.

We now turn to the accuracy of the numerical methods. We will compute the relative error in the L_2 and L_∞ norm. Recall that L_2 error is computed as:

$$\epsilon_2 = \left(\sum_i |V - V_i|^2 \right)^{1/2}$$

And the L_∞ error is computed as:

$$\epsilon_\infty = \max_i |V - V_i|$$

where V represents the actual price of the European vanilla option under fixed α and V_i represents the numerical value evaluated at x_i for $T = 1$. The results are presented below:

Table 5.2: Accuracy of Numerical Methods

m	ϵ_2 for FV	ϵ_∞ for FV
2^9	1.0531×10^{-3}	3.1012×10^{-3}
2^{10}	5.4532×10^{-4}	1.5926×10^{-3}
2^{11}	2.739×10^{-4}	7.1038×10^{-4}
2^{12}	1.297×10^{-4}	3.2579×10^{-4}

As expected, we note that as m increases, $\epsilon_2, \epsilon_\infty \to 0$, since we require finer temporal and spatial partitions in the simulation of the Black-Scholes pricing formula under the FMLS process. Also, note that under the finite volume scheme, our simulation may motivate more methods to efficiently compute the desired option prices given parameter α since these errors are insignificant to the application of option pricing.

5.2 Concluding Remarks

In this thesis, we have successfully analyzed the behavior of the Black-Scholes model under the FMLS process for varied choices of α, determined a closed-form solution to represent a variant of the Black-Scholes pricing formula in terms of α, and used numerical methods to simulate the recent behavior of the S&P 500 options market in a one-year time frame. We have also considered finite volume schemes to further reduce the CPU time by decreasing the computational and storage costs of the standard finite difference method, using banded coefficient matrices to efficiently compute the solution of the system. Lastly, we performed an applicable simulation emphasizing the efficiency of the finite volume method and have successfully achieved our desired results.

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Appendix A

Diffusion Models and Random Walks

In this Appendix, we provide a mathematically justified proof of the diffusion equation using applications in random walks. We will begin by proposing the Central Limit Theorem. We will provide no proof of the Central Limit Theorem in this Appendix.

Theorem A.1. *(Central Limit Theorem) Let $X_1, \cdots, X_n$ be independent, identically distributed random variables satisfying $E(X_i) = m$ and $Var(X_i) = \sigma^2 > 0$. Let $S_n := X_1 + \cdots + X_n$. Then for $a, b \in \mathbb{R}$, $a < b$,*

$$\lim_{n\to\infty} P\left(a \leq \frac{S_n - nm}{\sqrt{n}\sigma} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx.$$

In our mathematical justification of the diffusion model, we require the following theorem. We will provide a proof of the theorem.

Theorem A.2. *(Laplace DeMoivre Theorem) Let $X_1, \cdots, X_n$ be independent, identically distributed random variables satisfying,*

$$P(X_i = 1) = p \text{ and } P(X_i = 0) = q, \text{ where } i \in \{1, \cdots, n\}$$

for $p, q \geq 0$ and $p + q = 1$. Define $S_n := X_1 + \cdots X_n$. Then for $a, b \in \mathbb{R}$, $a < b$,

$$\lim_{n\to\infty} P\left(a \leq \frac{S_n - np}{\sqrt{npq}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx.$$

Remark: One can easily compute that $E(S_n) = np$ and $Var(S_n) = npq$. The Laplace-DeMoivre Theorem is a specific case of the Central Limit Theorem.

Proof. Define $S_n^* := \dfrac{S_n - np}{\sqrt{npq}}$ to be a random variable with value $x_k = \dfrac{k-np}{\sqrt{npq}}$ and probability $p_n(k) = \dbinom{n}{k} p^k q^{n-k}$. Then, for $a < b$,

$$P(a \leq S_n^* \leq b) = \sum_{a \leq x_k \leq b} p_n(k) = \sum_{a \leq x_k \leq b} \binom{n}{k} p^k q^{n-k} = \sum_{a \leq x_k \leq b} \frac{n!}{k!(n-k)!} p^k q^{n-k}.$$

By applying Stirling's Formula, which states that as $n \to \infty$:

$$n! = e^{-n} n^n \sqrt{2\pi n}(1 + o(1))$$

we have that as $n \to \infty$,

$$\begin{aligned} P(a \leq S_n^* \leq b) &= \sum_{a \leq x_k \leq b} \frac{e^{-n} n^n \sqrt{2\pi n} p^k q^{n-k}}{e^{-k} k^k \sqrt{2\pi k} e^{-(n-k)} (n-k)^{n-k} \sqrt{2\pi(n-k)}} (1 + o(1)) \\ &= \sum_{a \leq x_k \leq b} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} (1 + o(1)). \end{aligned}$$

Now we propose the following lemma:

Lemma A.3. *We have the following:*

$$\lim_{n \to \infty; \frac{k-np}{\sqrt{npq}} \to x} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} = e^{-\frac{x^2}{2}}$$

Proof. Let $x = x_k = \dfrac{k-np}{\sqrt{npq}}$. Note that,

$$1 + \sqrt{\frac{q}{np}} x = 1 + \sqrt{\frac{q}{np}} \left(\frac{k-np}{\sqrt{npq}}\right) = \frac{k}{np}$$

and,

$$1 + \sqrt{\frac{p}{nq}} x = 1 + \sqrt{\frac{p}{nq}} \left(\frac{k-np}{\sqrt{npq}}\right) = \frac{n-k}{nq}.$$

Given that as $y \to 0$, we have $\log(1 \pm y) = \pm y - \frac{y^2}{2} + O(y^3)$. Therefore, we have

$$\begin{aligned} \log\left(\frac{np}{k}\right)^k &= -k \log\left(\frac{k}{np}\right) \\ &= -k \log\left(1 + \sqrt{\frac{q}{np}} x\right) \\ &= -(np + x\sqrt{npq}) \left(\sqrt{\frac{q}{np}} x - \frac{q}{2np} x^2\right) + O\left(n^{-\frac{1}{2}}\right) \end{aligned}$$

And similarly,

$$\log\left(\frac{nq}{n-k}\right)^{n-k} = -(n-k)\log\left(\frac{n-k}{nq}\right)$$
$$= -(n-k)\log\left(1+\sqrt{\frac{p}{nq}}x\right)$$
$$= -(nq - x\sqrt{npq})\left(-\sqrt{\frac{p}{nq}}x - \frac{p}{2nq}x^2\right) + O\left(n^{-\frac{1}{2}}\right)$$

Adding the expressions and simplifying yields,

$$\lim_{n\to\infty;\frac{k-np}{\sqrt{npq}}\to x} \log\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k} = -\frac{x^2}{2}.$$

This implies the statement of the lemma. The lemma is proven. □

Again, let $x = x_k = \dfrac{k-np}{\sqrt{npq}}$. Therefore $k = np + x\sqrt{npq}$ and $n-k = nq - x\sqrt{npq}$. Thus,

$$\sqrt{\frac{n}{k(n-k)}} = \frac{1}{\sqrt{npq}}(1+o(1)).$$

Continuing from our calculations, as $n \to \infty$, we have:

$$P(a \le S_n^* \le b) = \frac{1}{\sqrt{2\pi}} \sum_{a\le x_k \le b} \frac{1}{\sqrt{npq}} e^{-\frac{x^2}{2}}(1+o(1)).$$

Note that the right hand side is simply a Riemann sum approximation as $n \to \infty$ of the following integral:

$$P(a \le S_n^* \le b) = \frac{1}{2\pi}\int_a^b e^{-\frac{x^2}{2}} dx.$$

This completes the proof of the theorem. □

We will now proceed to derive the traditional diffusion model using random walks. Given spacing $\Delta x > 0$ and time duration $\Delta t > 0$, define the following two-dimensional lattice:

$$\{(m\Delta x, n\Delta t) \mid m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}.$$

Consider a particle in starting position $x = 0$ at $t = 0$. We allow the particle to only move to the left and to the right. Let the probability of the particle moving left an amount of Δx be $\frac{1}{2}$. Therefore, the probability of the particle moving right Δx units is $\frac{1}{2}$.

Let $p(m, n)$ be the probability of a particle at position $m\Delta x$ and $n\Delta t$. Then, we have,

$$p(m,0) = \begin{cases} 0 & m \neq 0 \\ 1 & m = 0 \end{cases}$$

Furthermore, we have

$$p(m, n+1) = \frac{1}{2}p(m-1, n) + \frac{1}{2}p(m+1, n)$$

and equivalently,

$$p(m, n+1) - p(m, n) = \frac{1}{2}[p(m-1, n) - 2p(m, n) + p(m-1, n)]$$

Assume that $\dfrac{(\Delta x)^2}{\Delta t} = \sigma^2 > 0$. Then we have,

$$\frac{p(m, n+1) - p(m, n)}{\Delta t} = \frac{\sigma^2}{2}\frac{p(m-1, n) - 2p(m, n) + p(m-1, n)}{(\Delta x)^2}$$

Letting $\Delta t \to 0$, $\Delta x \to 0$, $m\Delta x \to x$, and $n\Delta t \to t$, we see that $p(m, n) \to u(x, t)$ where $u(x, t)$ is the probability density function of a particle at position x and time t. We also see that by passing limits, the difference equation becomes the traditional diffusion model, i.e.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}.$$

Now we will solve the traditional diffusion model by using another interpretation of random walks. Again, consider the particle with the same conditions previously mentioned. Let $X(t)$ be the position of the particle at time $t = n\Delta t$, and define

$S_n := \sum_{i=1}^{n} X_i$ where X_i are independent random variables satisfying $P(X_i = 0) = \frac{1}{2}$ and $P(X_i = 1) = \frac{1}{2}$. Note that $Var(X_i) = \frac{1}{4}$.

S_n is a random walk that determines the number of moves to the right at time $t = n\Delta t$. Therefore we have,

$$X(t) = S_n \Delta x + (n - S_n)(-\Delta x) = (2S_n - n)\Delta x.$$

Note that,

$$\begin{aligned} Var(X(t)) &= (\Delta x)^2 Var[(2S_n - n)] \\ &= 4(\Delta x)^2 Var[S_n] = 4n(\Delta x)^2 Var[X_i] \\ &= n(\Delta x)^2 = \frac{(\Delta x)^2}{\Delta t} t = \sigma^2 t. \end{aligned}$$

Continuing from our calculations, we have

$$X(t) = (2S_n - n)\Delta x = \left(\frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \right) \sqrt{n}\Delta x = \left(\frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \right) \sqrt{\sigma^2 t}.$$

Applying Theorem A.2 yields the following result:

$$\begin{aligned} \lim_{n\to\infty} P(a \le X(t) \le b) &= \lim_{n\to\infty} \left(\frac{a}{\sqrt{\sigma^2 t}} \le \frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \le \frac{b}{\sqrt{\sigma^2 t}} \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{\sigma^2 t}}}^{\frac{b}{\sqrt{\sigma^2 t}}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_a^b e^{-\frac{x^2}{2\sigma^2 t}} dx. \end{aligned}$$

This implies that the probability density function $u(x,t)$ satisfies

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t} \right).$$

The above solution also solves the traditional diffusion model.

Appendix B

Stochastic Calculus

We will begin Appendix B with a few necessary definitions that play a crucial role in the derivation of the Black-Scholes model.

Definition B.1. *A real-valued stochastic process* $W(\cdot)$ *defined on some probability space* $(\Omega, \mathcal{U}, P)$*, is called* ***Brownian motion*** *(or* ***Wiener process****) if* $W(\cdot)$ *satisfies the following conditions:*

- $W(0) = 0$ *almost surely,*
- $W(t) - W(s)$ *is* $N(0, t-s)$ *for all* $t \geq s \geq 0$*,*
- *for all times* $0 < t_1 < t_2 < \ldots < t_n$*, the random variables* $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ *are independent.*

The stochastic process W is imperative in reflecting how stock option market pricing behaves under the most simple conditions; namely, it attributes to the behavior's resemblance of a random walk which inspires the construction of the traditional diffusion model. We will derive the diffusion model later in this section.

Definition B.2. *Let* W *be a Wiener process. For times* $0 \leq t \leq T$*, we define* ***Ito's Integral*** *to be the following:*

$$\int_0^T W dW := \frac{W^2(T)}{2} - \frac{T}{2}.$$

We will assume that all stochastic integrals, including the Ito's Integral, exists and are well defined throughout this paper. More details on the existence of stochastic integrals can be referred to Evans. [?Evans13] We propose the following definition:

Definition B.3. *Let $X(\cdot)$ be a real-valued stochastic process satisfying*

$$X(r) = X(s) + \int_s^r \mu dt + \int_s^r \sigma dW$$

where μ and σ are real-valued progressively measurable processes satisfying

$$\mathbb{E}\left(\int_0^T |\mu| dt\right) < \infty \text{ and } \mathbb{E}\left(\int_0^T \sigma^2 dt\right) < \infty$$

and s and r are times satisfying $0 \leq s \leq r \leq T$. Then for $0 \leq t \leq T$, we say that $X(\cdot)$ an ***Ito process*** *with* ***stochastic differential*** *$dX = \mu dt + \sigma dW$ for* ***drift*** *μ and* ***volatility*** *σ.*

We will begin this section by proposing Ito's Chain Rule.

Theorem B.1. *(Ito's Chain Rule) Let $f(x,t)$ be a smooth function and let $X(t)$ be an Ito process with stochastic differential $dX = \mu dt + \sigma dW$. Then $Y(t) := f(X(t), t)$ is also an Ito process with stochastic differential:*

$$dY = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 dt$$

Or equivalently,

$$dY = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dW.$$

Ito's Chain Rule plays a crucial role in the stochastic derivation of the Black-Scholes Equation. In order to prove Theorem B.1, we need to propose and prove Ito's Product Rule.

Theorem B.2. *(Ito's Product Rule) Let $X_1(t)$ and $X_2(t)$ be Ito processes satisfying $dX_1 = \mu_1 dt + \sigma_1 dW$ and $dX_2 = \mu_2 dt + \sigma_2 dW$. Suppose that μ_1, μ_2, σ_1, and σ_2 are real-valued progressively measureable processes satisfying,*

$$\mathbb{E}\left(\int_0^T |\mu_1| dt\right), \mathbb{E}\left(\int_0^T |\mu_2| dt\right) < \infty \text{ and } \mathbb{E}\left(\int_0^T \sigma_1^2 dt\right), \mathbb{E}\left(\int_0^T \sigma_2^2 dt\right) < \infty$$

for $0 \leq t \leq T$*. Then,* $X_1X_2(t)$ *is an Ito process satisfying*

$$d(X_1X_2) = X_2dX_1 + X_1dX_2 + \sigma_1\sigma_2 dt.$$

Remark: Note that the integrated version of Ito's product rule yields **Ito integration-by-parts formula**:

$$\int_s^r X_2dX_1 = X_1(r)X_2(r) - X_1(s)X_2(s) - \int_s^r X_1dX_2 - \int_s^r \sigma_1\sigma_2 dt.$$

Proof. Choose $0 \leq r \leq T$. Let $\mathcal{F}(t) := \mathcal{U}(W(s), X_0)$ be the $\sigma-$algebra generated by X_0 and $W(s)$ for $0 \leq s \leq t$, where X_0 is a random variable independent of the future of Brownian motion beyond time $t = 0$. Firstly, assume for simplicity that $X_1(0) = X_2(0) = 0$, $\mu_1(t) = \mu_1$, $\mu_2(t) = \mu_2$, $\sigma_1(t) = \sigma_1$, and $\sigma_2(t) = \sigma_2$, where μ_1, μ_2, σ_1, and σ_2 are time independent, $\mathcal{F}(0)-$measurable random variables. Then for $t \geq 0$, we have $X_1(t) = \mu_1 t + \sigma_1 W(t)$ and $X_2(t) = \mu_2 t + \sigma_2 W(t)$.

Thus,

$$\begin{aligned}
\int_0^r X_2dX_1 + X_1dX_2 + \sigma_1\sigma_2 dt &= \int_0^r X_1\mu_2 + X_2\mu_1 dt + \int_0^r X_1\sigma_2 + X_2\sigma_1 dW + \int_0^r \sigma_1\sigma_2 dt \\
&= \int_0^r (\mu_1 t + \sigma_1 W)\mu_2 + (\mu_2 t + \sigma_2 W)\mu_1 dt \\
&\quad + \int_0^r (\mu_1 t + \sigma_1 W)\sigma_2 + (\mu_2 t + \sigma_2 W)\sigma_1 dW + \int_0^r \sigma_1\sigma_2 dt \\
&= \mu_1\mu_2 r^2 + (\sigma_1\mu_2 + \sigma_2\mu_1)\left(\int_0^r W dt + \int_0^r t dW\right) \\
&\quad + 2\sigma_1\sigma_2 \int_0^r W dW + \sigma_1\sigma_2 r.
\end{aligned}$$

By Definition B.2, we have $2\int_0^r W dW = W^2(r) - r$. We propose and prove the following lemma:

Lemma B.3. *For* $r \geq 0$, $\int_0^r t dW + \int_0^r W dt = rW(r)$.

Proof. Let $P^n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = r\}$ be a sequence of partitions of the interval $[0, r]$ with $|P^n| \to 0$ as $n \to \infty$. Therefore, by definition of the stochastic integral using Riemann sum approximation, we have:

$$\int_0^r tdW = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} t_k^n(W(t_{k+1}^n) - W(t_k^n)).$$

Similarly, since $t \mapsto W(t)$ is continuous almost surely, we have,

$$\int_0^r Wdt = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} W(t_{k+1}^n)(t_{k+1}^n - t_k^n).$$

Therefore, we have,

$$\begin{aligned}
\int_0^r tdW + \int_0^r Wdt &= \lim_{n\to\infty} \sum_{k=0}^{m_n-1} t_k^n(W(t_{k+1}^n) - W(t_k^n)) + W(t_{k+1}^n)(t_{k+1}^n - t_k^n) \\
&= \lim_{n\to\infty} \sum_{k=0}^{m_n-1} t_{k+1}^n W(t_{k+1}^n) - t_k^n W(t_k^n) \\
&= \lim_{n\to\infty} t_{m_n}^n W(t_{m_n}^n) - t_0^n W(t_0^n) \\
&= rW(r)
\end{aligned}$$

□

With these identities, we have

$$\begin{aligned}
\int_0^r X_2 dX_1 + X_1 dX_2 + \sigma_1\sigma_2 dt &= \mu_1\mu_2 r^2 + (\sigma_1\mu_2 + \sigma_2\mu_1) rW(r) + \sigma_1\sigma_2 W^2(r) \\
&= X_1(r)X_2(r)
\end{aligned}$$

which is equivalent to the Ito integration-by-parts formula for the specific case of $s = 0$, $X_1(0) = X_2(0) = 0$, and μ_1, μ_2, σ_1 and σ_2 are time-independent random variables. A similar proof can be extended for the case of $s \geq 0$, $X_1(s)$ and $X_2(s)$ are arbitrary, and μ_1, μ_2, σ_1 and σ_2 are constant $\mathcal{F}(s)$-measurable random variables. Ito's Product Rule holds for constant random variables.

Suppose that μ_1, μ_2, σ_1, and σ_2 are step processes. For each subinterval $[t_k, t_{k+1})$ for which μ_1, μ_2, σ_1, and σ_2 are constant random variables, we can apply the previous steps and add the resulting integrals. Ito's Product Rule holds for step processes.

Now suppose that μ_1, μ_2, σ_1, and σ_2 are general processes. We can select a sequence of step processes $\mu_1^n, \mu_2^n, \sigma_1^n$, and σ_2^n such that

$$E\left(\int_0^T |\mu_1^n - \mu_1| dt\right) \to 0 \text{ and } E\left(\int_0^T |\mu_2^n - \mu_2| dt\right) \to 0$$

and

$$E\left(\int_0^T (\sigma_1^n - \sigma_1)^2 dt\right) \to 0 \text{ and } E\left(\int_0^T (\sigma_2^n - \sigma_2)^2 dt\right) \to 0$$

holds. Define

$$X_1^n(t) := X_1(0) + \int_0^t \mu_1^n ds + \int_0^t \sigma_1^n dW$$
$$X_2^n(t) := X_2(0) + \int_0^t \mu_2^n ds + \int_0^t \sigma_2^n dW$$

Note that X_1^n and X_2^n have stochastic differentials $dX_1^n = \mu_1^n dt + \sigma_1^n dW$ and $dX_2^n = \mu_2^n dt + \sigma_2^n dW$. Since X_1^n and X_2^n are step processes, Ito's product rule applies. This yields $d(X_1^n X_2^n) = X_2^n dX_1^n + X_1^n dX_2^n + \sigma_1^n \sigma_2^n dt$. Letting $n \to \infty$ yields the statement of the theorem. □

Proof. Let X be an Ito process with stochastic differential $dX = \mu dt + \sigma dW$. Firstly, consider the case $f(x,t) = x^m$ for $m \in \mathbb{N}$. We claim that

$$d(X^m) = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}\sigma^2 dt$$

and prove the claim with induction on m. Letting $m = 0, 1$ yields trivial cases and for $m = 2$, we have $d(X^2) = 2XdX + \sigma^2 dt$. This ie equivalent to the statement of Theorem 1.2 for $X_1 = X_2 = X$. Let us assume that

$$\begin{aligned} d(X^{m-1}) &= (m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}\sigma^2 dt \\ &= (m-1)X^{m-2}(\mu dt + \sigma dW) + \frac{1}{2}(m-1)(m-2)X^{m-3}\sigma^2 dt. \end{aligned}$$

Then, using Theorem B.2, we have

$$\begin{aligned}
d(X^m) &= d(XX^{m-1}) \\
&= Xd(X^{m-1}) + X^{m-1}dX + (m-1)X^{m-2}\sigma^2 dt \\
&= X\left((m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}\sigma^2 dt\right) \\
&\quad + X^{m-1}dX + (m-1)X^{m-2}\sigma^2 dt \\
&= mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}\sigma^2 dt
\end{aligned}$$

The induction is complete. We can generalize the above case for all polynomials $f(x,t)$ in the variable x, because the stochastic differential operator is linear. Ito's Chain Rule is therefore proven for all polynomials $f(x,t)$ in the variable x.

Now suppose that $f(x,t) = f_1(x)f_2(t)$ for polynomials f_1 and f_2. Then,

$$\begin{aligned}
d(f(X,t)) &= d(f_1(X)f_2) \\
&= f_1(X)df_2 + f_2 df_1(X) \\
&= f_1(X)f_2' dt + f_2\left(f_1'(X)dX + \frac{1}{2}f_1''(X)\sigma^2 dt\right) \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt
\end{aligned}$$

By the above calculations, Ito's Chain Rule applies for polynomials of the form $f(x,t) = f_1(x)f_2(t)$. This can be generalized for all polynomial functions f of the variables t and x since any f can be re-expressed as a linear combination of functions of the form $f(x,t) = f_1(x)f_2(t)$ and the stochastic differential operator is linear.

Now let $f(x,t)$ be a smooth function. There exists a sequence of polynomials f^n such that $f^n \to f$, $\frac{\partial f^n}{\partial t} \to \frac{\partial f}{\partial t}$, $\frac{\partial f^n}{\partial x} \to \frac{\partial f}{\partial x}$, and $\frac{\partial^2 f^n}{\partial x^2} \to \frac{\partial^2 f}{\partial x^2}$. By applying the previous steps, we can deduce that for all $0 \le s \le T$,

$$f^n(s, X(s)) - f^n(0, X(0)) = \int_0^s \frac{\partial f^n}{\partial t} + \frac{\partial f^n}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f^n}{\partial x^2}\sigma^2 dt + \int_0^s \frac{\partial f^n}{\partial x}\sigma dW \text{ a.s.};$$

Letting $n \to \infty$ yields Ito's Chain Rule for all smooth functions $f(x,t)$. □

Appendix C

Black-Scholes Formula

In this Appendix, we will transform the Black-Scholes equation into the traditional diffusion equation by making specific substitutions. We have determined the solution of the diffusion equation in Appendix A. Finally, we will use this solution to derive the Black-Scholes Formula mentioned in Chapter 1.

Let $C(S,t)$ be the price of a European call option with asset price S and time t. Recall that the Black-Scholes model satisfies the following differential equation:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

and boundary conditions that for $0 \leq t \leq T$ and $0 < S < \infty$, we have $C(S,t) \sim S$ as $S \to \infty$, and $C(S,T) = \max\{S - K, 0\}$ for strike price K.

Firstly, let $S = Ke^x$, $t = T - \frac{2}{\sigma^2}\tau$, and $C(S,t) = Kv(x,t)$. Then, we have $x = \log(S/K)$ and $\tau = \frac{1}{2}\sigma^2(T-t)$.

Equivalently, we have $d\tau = -\frac{1}{2}\sigma^2 dt$ and $\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x}$. Thus,

$$\frac{\partial^2}{\partial S^2} = \frac{\partial}{\partial S}\frac{\partial}{\partial S} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S^2}\frac{\partial^2}{\partial x^2}.$$

Making the above substitutions to the Black-Scholes equation yields,

$$-\frac{1}{2}\sigma^2\frac{\partial v}{\partial \tau} + rS\left[\frac{1}{S}\frac{\partial v}{\partial x}\right] + \frac{1}{2}\sigma^2 S^2\left[-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right] = rv.$$

Or equivalently,

$$\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} - \frac{2r}{\sigma^2}\frac{\partial v}{\partial x} + \frac{2r}{\sigma^2}v = 0.$$

Redefining $k := \dfrac{2r}{\sigma^2}$ and rearranging yields,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv.$$

Under these transformations, the boundary condition of $C(S,T) = \max\{S-K, 0\}$ for strike price K becomes $v(x,0) = \max\{e^x - 1, 0\}$.

We now make the following substitution to further transform the above differential equation into a diffusion equation. Let $v(x,\tau) = \gamma u(x,\tau)$ where $\gamma(x,\tau) = \exp(\alpha x + \beta\tau)$. Then, we have the following:

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \beta\gamma(x,\tau)u + \gamma(x,\tau)\frac{\partial u}{\partial \tau} \\
\frac{\partial v}{\partial x} &= \alpha\gamma(x,\tau)u + \gamma(x,\tau)\frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2\gamma(x,\tau)u + 2\alpha\gamma(x,\tau)\frac{\partial u}{\partial x} + \gamma(x,\tau)\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Making this substitution to the above differential equation and dividing both sides by $\gamma(x,\tau)$ yields,

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

Equivalently, we have,

$$\frac{\partial u}{\partial \tau} = (-\beta + \alpha^2 + \alpha(k-1) - k)u + (2\alpha + k - 1)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}.$$

We require that $\alpha = \dfrac{1}{2}(1-k)$ and $\beta = -\dfrac{1}{4}(1+k)^2$ to force the above partial differential equation to become a diffusion model. With these choices of parameters, we have,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

with $u_0(x) := u(x,0) = \max\left\{\exp\left(\tfrac{1}{2}(k+1)x\right) - \exp\left(\tfrac{1}{2}(k-1)x\right), 0\right\}$.

The diffusion equation that we have derived in the paper and Appendix A all have boundary conditions independent of x. We generalize the solution of the diffusion equation by applying the following definition:

Definition C.1. *Let $u(x,t)$ satisfy the following diffusion model:*

$$\frac{\partial u}{\partial \tau} = k\frac{\partial^2 u}{\partial x^2}.$$

for some constant k and boundary condition $u(x,0) = g(x)$. Then, ***Green's function for the diffusion model*** *is given as follows:*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) g(y)dy.$$

Applying Definition C.1 for $k = 1$ and the change of variables $y \mapsto x + \sqrt{2\tau}y$, we see that

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(x + \sqrt{2\tau}y) \exp\left(-\frac{y^2}{2}\right) dy.$$

Reverting the solution to the original variables yields the desired Black-Scholes formula given as follows:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where,

$$N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}s^2\right) ds$$

And,

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \end{aligned}$$

More details can be referred to Dewynne, Howison, and Wilmott. [?Dewynne95]

Appendix D

Fourier Transforms of Diffusion Models

We will begin this Appendix by providing the statement of the Levy Continuity Theorem:

Theorem D.1. *(Levy Continuity Theorem) Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random variables with characteristic functions* φ_n*. If* $\varphi_n \to \varphi$ *converges point-wise, the following statements are equivalent,*

- X_n *converges in distribution to some* X
- $\{X_n\}_{n=1}^{\infty}$ *is tight, i.e.* $\lim_{x\to\infty}\left(\sup_n \mathbb{P}[|X_n| > x]\right) = 0$
- φ *is a characteristic function of some random variable* X
- φ *is a continuous function*
- φ *is continuous in some neighborhood of 0.*

A proof of the theorem can be referred to Fristedt and Gray.

We will provide the mathematical details of the inversion of Fourier transforms to yield their respective diffusion models. We will make use of the Fourier Inversion Theorem mentioned below:

Theorem D.2. *(Fourier Inversion Theorem) If* $\int |f(x)|dx < \infty$*, then the Fourier transform* $\hat{f}(k)$ *exists. If* $|\hat{f}(k)|dk < \infty$*, then for all* $x \in \mathbb{R}$*,*

$$f(x) = \frac{1}{2\pi}\int e^{ikx}\hat{f}(k)dk.$$

We will not prove this theorem in this paper. Recall that in this paper, we have derived the following differential equation:

$$\frac{d\hat{u}(k,t)}{dt} = (ik)^\alpha \hat{u}(k,t).$$

Applying Theorem D.2 on both sides yields,

$$\frac{1}{2\pi}\int e^{ikx}\frac{\partial}{\partial t}\hat{u}(k,t)dk = \frac{1}{2\pi}\int e^{ikx}(ik)^\alpha \hat{u}(k,t)dk$$

Example 1.1 and Theorem D.2 combined implies that,

$$\frac{1}{2\pi}\int e^{ikx}(ik)^\alpha \hat{u}(k,t)dk = \frac{\partial^\alpha}{\partial x^\alpha}u(x,t).$$

Since,

$$\frac{\partial}{\partial t}\frac{1}{2\pi}\int e^{ikx}\hat{u}(k,t)dk = \frac{\partial}{\partial t}u(x,t),$$

it suffices to show that for all fixed $t > 0$, the following holds,

$$\int e^{ikx}\frac{\partial}{\partial t}\hat{u}(k,t)dk = \frac{\partial}{\partial t}\int e^{ikx}\hat{u}(k,t)dk.$$

for the Fourier transform $\hat{u}(k,t) = e^{t(ik)^\alpha}$ of a stable density. Note that,

$$\frac{\partial}{\partial t}\int e^{ikx}\hat{u}(k,t)dk = \lim_{h\to 0}\int e^{ikx}\frac{\hat{u}(k,t+h)-\hat{u}(k,t)}{h}dk$$

and,

$$\left|\frac{\hat{u}(k,t+h)-\hat{u}(k,t)}{h}\right| = \left|e^{t(ik)^\alpha}\right|\left|\frac{1-e^{h(ik)^\alpha}}{h(ik)^\alpha}\right||(ik)^\alpha|.$$

Note that we have,

$$(ik)^\alpha = (i\text{sgn}(k)|k|)^\alpha = |k|^\alpha \exp\left(i\text{sgn}(k)\frac{\pi\alpha}{2}\right) = |k|^\alpha\left(\cos\frac{\pi\alpha}{2} + i\text{sgn}(k)\sin\frac{\pi\alpha}{2}\right)$$

Thus,

$$\left|e^{t(ik)^\alpha}\right| = e^{\text{Re}(t(ik)^\alpha)} = e^{t|k|^\alpha\cos\frac{\pi\alpha}{2}}$$

and, $|(ik)^\alpha| = |k|^\alpha$. Furthermore, note that by Taylor expansion, we have,

$$\left|\frac{1-e^z}{z}\right| \leq 1 + \frac{|z|}{2!} + \frac{|z|^2}{3!} + \cdots = \frac{e^{|z|}-1}{|z|}.$$

For fixed $t > 0$ and $z = h(ik)^\alpha$ such that $|h| < -\frac{t}{2}\left(\cos\frac{\pi\alpha}{2}\right)$, by the mean value theorem,

$$e^{|h||k|^\alpha} - 1 \leq |h||k|^\alpha \exp\left(-|k|^\alpha \frac{t}{2}\cos\frac{\pi\alpha}{2}\right).$$

Thus,

$$\left|\frac{1-e^{h(ik)^\alpha}}{h(ik)^\alpha}\right| \leq \frac{e^{|h||k|^\alpha}-1}{|h||k|^\alpha} \leq \exp\left(-|k|^\alpha \frac{t}{2}\cos\frac{\pi\alpha}{2}\right).$$

Combining the three terms yields,

$$\left|\frac{\hat{u}(k,t+h)-\hat{u}(k,t)}{h}\right| \leq |k|^\alpha \exp\left(|k|^\alpha \frac{t}{2}\cos\frac{\pi\alpha}{2}\right).$$

Note that for any $t > 0$, the function $|k|^\alpha \exp\left(|k|^\alpha \frac{t}{2}\cos\frac{\pi\alpha}{2}\right)$ is integrable with respect to k. A variation of the Dominated Convergence Theorem is given as follows. We will not provide proof of the theorem.

Theorem D.3. *(Dominated Convergence Theorem) Let $f_n(x)$ be a sequence of functions satisfying $f_n(x) \to f(x)$ as $n \to \infty$. If $f_n(x) \leq g(x)$ for all x and n, and if $g(x)$ is integrable, then $\int f_n(x)dx \to \int f(x)dx$ and the integrals exists.*

Applying Theorem D.3 yields the following:

$$\frac{\partial}{\partial t}\int e^{ikx}\hat{u}(k,t)dk = \int e^{ikx}\lim_{h\to 0}\frac{\hat{u}(k,t+h)-\hat{u}(k,t)}{h}dk = \int e^{ikx}\frac{\partial}{\partial t}\hat{u}(k,t)dk.$$

This argument holds for all α contained in the interval $1 < \alpha \leq 2$. Applying this argument for $\alpha = 2$ yields the second-order partial derivative case.